Geometric control condition for the wave equation with a time-dependent observation domain

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Abstract

We characterize the observability property (and, by duality, the controllability and the stabilization) of the wave equation on a Riemannian manifold Ω, with or without boundary, where the observation (or control) domain is time-varying. We provide a condition ensuring observability, in terms of propagating bicharacteristics. This condition extends the well-known geometric control condition established for fixed observation domains.

As one of the consequences, we prove that it is always possible to find a time-dependent observation domain of arbitrarily small measure for which the observability property holds. From a practical point of view, this means that it is possible to reconstruct the solutions of the wave equation with only few sensors (in the Lebesgue measure sense), at the price of moving the sensors in the domain in an adequate way.

We provide several illustrating examples, in which the observation domain is the rigid displacement in Ω of a fixed domain, with speed v, showing that the observability property depends both on v and on the wave speed. Despite the apparent simplicity of some of our examples, the observability property can depend on nontrivial arithmetic considerations.

Keywords: wave equation, geometric control condition, time-dependent observation domain.
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Contents

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1 Introduction and main result

1.1 Framework

Studies of the stabilization and the controllability for the wave equation go back to the 70's with the works of D. L. Russell (see, e.g., [19, 20]). The work of J.-L. Lions was very important in the formalization of many controllability questions (see, e.g., [13]). In the case of a manifold without boundary Ω, the pioneering work of J. Rauch and M. Taylor [18] related the question of fast stabilization, that is, exhibiting an exponential decay of the energy, to a geometric condition connecting the damping region $\omega \subset \Omega$ and the rays of geometrical optics, the now celebrated Geometric Control Condition (in short, GCC). The damped wave equation takes the form

$$\partial_t^2 u - \triangle u + \chi_\omega \partial_t u = 0.$$

Using that the energy of the solution to a hyperbolic equation is largely carried along the rays, if one assumes that any ray will have reached the region ω where the operator is dissipative in a finite time, one can prove that the energy decays exponentially in time, with an additional unique continuation argument that allows one to handle the low frequency part of the energy. The work [18] only treated the case of a manifold Ω without boundary, leaving open the case of manifolds with boundary until the work of C. Bardos, G. Lebeau, and J. Rauch [2]. The understanding of the propagation of singularities in the presence of the boundary $\partial\Omega$, after the seminal work of R. Melrose and J. Sjöstrand [15, 16], was a key element in the proof of [2], providing a generalized notion of rays, taking reflections at the boundary into account as well as glancing and gliding phenomena. The geometric condition for ω, now an open subset of $\overline{\Omega}$, is then the requirement that every generalized ray should meet the damping region ω in a finite time. The resulting stabilization estimate then takes the form:

$$E_0(u(t)) \leqslant Ce^{-C't} E_0(u(0)),$$

where E_0 is the following energy

$$E_0(u(t)) = \|u(t)\|^2_{H^1(\Omega)} + \|\partial_t u(t)\|^2_{L^2(\Omega)}.$$

Note that, if an open set ω does not fulfill the geometric control condition, then only a logarithmic type of energy decay can be achieved in general [11, 12, 3].[1]

The question of exact controllability relies on the same line of arguments as for the exponential stabilization. By exact controllability in time $T > 0$, for the control wave equation

$$\partial_t^2 u - \triangle u = \chi_\omega(x) f,$$

one means, given an arbitrary initial state (u_0, u_1) and an arbitrary final state (u_0^F, u_1^F), the ability to find f such that $(u_{t=T}, \partial_t u_{t=T}) = (u_0^F, u_1^F)$ starting from $(u_{t=0}, \partial_t u_{t=0}) = (u_0, u_1)$. If the energy level is $(u(t), \partial_t u(t)) \in H^1(\Omega) \oplus L^2(\Omega)$, it is natural to seek $f \in L^2((0,T) \times \Omega)$. Boundary conditions can be of Dirichlet or Neumann types.

In fact, as is well known, both exponential stabilization and exact controllability of the wave equation in a domain Ω, with a damping or a control only acting in an open region ω of $\overline{\Omega}$, are equivalent to an observability estimate for a free wave. For such a wave, the energy is constant with respect to time. The observability inequality takes the following form: for some constant $C > 0$ and some $T > 0$, we have

$$E_0(u) \leqslant C \int_0^T \|\partial_t u\|^2_{L^2(\omega)} dt. \tag{1}$$

For the issue of exact controllability, the time $T > 0$ in this inequality is then the control time (horizon). If the open set ω fulfills the Geometric Control Condition, then the results of [18, 2] show that the infimum of all possible such times T coincides with the infimum of all possible times in the Geometric Control Condition. Note however that there are cases in which the Geometric Control Condition does not hold, and yet the observability inequality (1) is valid: the case Ω is a sphere and ω is a half-sphere is a typical example.

A glance at inequality (1), shows that observability is in fact to be understood as occuring in a space-time domain, here $(0,T) \times \omega$. It is then natural to wonder if observability can hold if it is replaced by some other open subset of $(0,T) \times \overline{\Omega}$. This is the subject of the present article.

The motivation for such a study can be seen as fairly theoretical. However, in practical issues, in different industrial contexts, for nondestructive testing, safety applications, as well as tomography techniques used for imaging bodies (human or not), this question becomes quite relevant. In fact, the industrial framework of seismic exploration was the original motivation for this work. In the different fields we mentionned, data are collected to be exploited in an interpretation step which involves the solution of some inverse problem. The point is that the device used to collect data does not fit well with the usual geometric condition which is crucial to obtain an observability result. In some cases it appears of great interest to be able to tackle situations where the observation set is time-dependent. In others, the reduction of data volume may be sought, while preserving the data quality. One may also face a situation in which all sensors cannot be active at the same time.

The example of seismic data acquisition can help the reader get a grasp on the industrial need to better design data acquisition procedures. In the case of a towed marine seismic data acquisition campaign, a typical setup consists in six parallel streamers with length 6000 m, separated by a distance of 100 m, floating at a depth of 8 m. The basic receiving elements are pressure sensitive hydrophones composed of piezoelectric ceramic crystal devices that are placed some 20 to 50 m apart along each streamer. A source (a carefully designed air gun array) is shot every 25 m while the boat moves. The seismic data experiment lasts around 8 s. One understands with this description that a huge amount (terabytes) of data is recorded during one such acquisition campaign above

[1] In fact, intermediate decay rates have been established in particular geometrical settings, see for instance [21] for almost exponential decay, [5, 17, 1] for polynomial decay.

an area of interest beneath the sea floor. Of course, the velocity of the ship and of the streamers is very small as compared to that of the seismic waves (1500 m/s in water and up to 5000 m/s for examples in salt bodies that are typical in the North Sea or in the Gulf of Mexico). Yet, however small it may be, one can question its impact on the quality of the data. One could also want not to use all receivers at a single time but rather to design a dynamic (software based) array of receivers during the time of the seismic experiment. The reader will of course realize that the mathematical results we present here are very far from solving this problem. They however give some leads on what important theoretical issues can be.

An inspection of the proof of [2] shows that it uses the invariance of the observation cylinder $(0,T)\times\omega$ with respect to time in a crucial way. Hence, the method, if not modified, cannot be applied to a general open subset of $(0,T)\times\overline{\Omega}$. One of the contributions of the present work is to remedy this issue. In fact, this is done by a significant simplification of the argument of [2], yielding a less technical aspect in one of the steps of the proof. Eventually, the result that we obtain is in fact faithful to the intuition one may have. The proper geometric condition to impose on an open subset Q of $(0,T)\times\overline{\Omega}$, for an observability condition of the form

$$E_0(u)\leqslant C\iint_Q |\partial_t u|^2\,dt\,dx \tag{2}$$

to hold is the following: for any generalized ray $t\mapsto x(t)$ initiated at time $t=0$, there should be a time $0<t_1<T$ such that the ray is located in $Q\cap\{t=t_1\}$, that is $(t_1,x(t_1))\in Q$. This naturally generalizes the usual Geometric Control Condition in the case where Q is the cylinder $(0,T)\times\omega$.

One of the interesting consequences of our analysis lies in the following fact: if the geometric condition holds for a time-dependent domain Q, a thinner domain for which the condition holds as well can be simply obtained by picking a neighborhood of the boundary of Q. This can be viewed as a step towards the reduction of the amount of data collected in the practical applications mentioned above.

We complete our analysis with a set of examples in *very simple* geometrical situations. Some of these examples show that even if "many" rays are missed by a static domain, a moving version of this domain can capture in finite time all rays, even with a very slow motion. However other examples show situations in which "very few" rays are missed, and a slow motion of the observation set allows one to capture these rays, yet implying that other rays remain away from the moving observation region for any positive time. Those examples may become hard to analyse because of the complexity of the Hamiltonian dynamics that governs the rays. Yet, they illustrate that naive strategies can fail to achieve the fulfillment of the Geometric Control Condition. Those examples show that further study would be of interest, with a study of the increase or decrease of the minimal control time as an observation set is moved around. Some examples show that this minimal control time may not be continuous with respect to the dynamics we impose on a moving control region.

1.2 Setting

Let (M,g) be a smooth d-dimensional Riemannian manifold, with $d\geqslant 1$. Let Ω be an open bounded connected subset of M, with a smooth boundary if $\partial\Omega\neq\emptyset$. We consider the wave equation

$$\partial_t^2 u-\triangle_g u=0, \tag{3}$$

in $\mathbb{R}\times\Omega$. Here, $\triangle_g$ denotes the Laplace-Beltrami operator on M, associated with the metric g on M. If the boundary $\partial\Omega$ of Ω is nonempty, then we consider boundary conditions of the form

$$Bu=0\quad\text{on } \mathbb{R}\times\partial\Omega, \tag{4}$$

where the operator B is either:

- the Dirichlet trace operator, $Bu = u_{|\partial\Omega}$;
- or the Neumann trace operator, $Bu = \partial_n u_{|\partial\Omega}$, where ∂_n is the outward normal derivative along $\partial\Omega$.

Our study encompasses the case where $\partial\Omega = \emptyset$: in this case, Ω is a compact connected d-dimensional Riemannian manifold. Measurable sets are considered with respect to the Riemannian measure dx_g (if M is the usual Euclidean space $\mathbb{R}^n$ then dx_g is the usual Lebesgue measure).

In the case of a manifold without boundary or in the case of homogeneous Neumann boundary conditions, the Laplace-Beltrami operator is not invertible on $L^2(\Omega)$ but is invertible in

$$L^2_0(\Omega) = \Big\{ u \in L^2(\Omega) \mid \int_\Omega u(x)\, dx_g = 0 \Big\}.$$

In what follows, we set $X = L^2_0(\Omega)$ in the boundaryless or in the Neumann case, and $X = L^2(\Omega)$ in the Dirichlet case (in both cases, the norm on X is the usual L^2-norm). We denote by $A = -\triangle_g$ the Laplace operator defined on X with domain $D(A) = \{u \in X \mid Au \in X \text{ and } Bu = 0\}$ with one of the above boundary conditions whenever $\partial\Omega \neq \emptyset$.

Note that A is a selfadjoint positive operator. In the case of Dirichlet boundary conditions, $X = L^2(\Omega)$ and we have $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$, $D(A^{1/2}) = H^1_0(\Omega)$ and $D(A^{1/2})' = H^{-1}(\Omega)$, where the dual is considered with respect to the pivot space X. For Neumann boundary conditions, $X = L^2_0(X)$ and we have $D(A) = \{u \in H^2(\Omega) \cap L^2_0(X) \mid \frac{\partial u}{\partial n}_{|\partial\Omega} = 0\}$ and $D(A^{1/2}) = H^1(\Omega) \cap L^2_0(X)$. The Hilbert spaces $D(A)$, $D(A^{1/2})$, and $D(A^{1/2})'$ are respectively endowed with the norms $\|u\|_{D(A)} = \|Au\|_{L^2(\Omega)}$, $\|u\|_{D(A^{1/2})} = \|A^{1/2}u\|_{L^2(\Omega)}$ and $\|u\|_{D(A^{1/2})'} = \|A^{-1/2}u\|_{L^2(\Omega)}$.

For all $(u^0, u^1) \in D(A^{1/2}) \times X$ (resp. $X \times D(A^{1/2})'$), there exists a unique solution $u \in \mathscr{C}^0(\mathbb{R}; D(A^{1/2})) \cap \mathscr{C}^1(\mathbb{R}; X)$ (resp., $u \in \mathscr{C}^0(\mathbb{R}; X) \cap \mathscr{C}^1(\mathbb{R}; D(A^{1/2})'))$ of (3)-(4) such that $u_{|t=0} = u^0$ and $\partial_t u_{|t=0} = u^1$. In both cases, such solutions of (3)-(4) are to be understood in a weak sense.

Remark 1.1. In (3), we consider the classical d'Alembert wave operator $\Box_g = \partial_t^2 - \triangle_g$. In fact, the results of the present article remain valid for the more general wave operators of the form

$$P = \partial_t^2 - \sum_{i,j} a_{ij}(x)\partial_{x_i}\partial_{x_j} + \text{lower-order terms},$$

where $(a_{ij}(x))$ is a smooth real-valued symmetric positive definite matrix, and where the lower-order terms are smooth and do not depend on t. We insist on the fact that our approach is limited to operators with time-independent coefficients as in [2].

1.3 Observability

Let Q be an open subset of $\mathbb{R} \times \overline{\Omega}$. We denote by χ_Q the characteristic function of Q, defined by $\chi_Q(t,x) = 1$ if $(t,x) \in Q$ and $\chi_Q(t,x) = 0$ otherwise. We set

$$\omega(t) = \{x \in \Omega \mid (t,x) \in Q\},$$

so that $Q = \{(t,x) \in \mathbb{R} \times \Omega \mid t \in \mathbb{R},\ x \in \omega(t)\}$. Let $T > 0$ be arbitrary. We say that (3)-(4) is observable on Q in time T if there exists $C > 0$ such that

$$C\|(u^0,u^1)\|^2_{D(A^{1/2})\times X} \leqslant \|\chi_Q \partial_t u\|^2_{L^2((0,T)\times\Omega)} = \int_0^T \int_{\omega(t)} |\partial_t u(t,x)|^2 \, dx_g \, dt, \tag{5}$$

for all $(u^0, u^1) \in D(A^{1/2}) \times X$, where u is the solution of (3)-(4) with initial conditions $u_{|t=0} = u^0$ and $\partial_t u_{|t=0} = u^1$. One refers to (5) as to an *observability inequality*.

The observability inequality (5) is stated for initial conditions $(u^0, u^1) \in D(A^{1/2}) \times X$. Other energy spaces can be used. An important example is the following proposition.

Proposition 1.2. *The observability inequality* (5) *is equivalent to having $C > 0$ such that*

$$C\|(u^0, u^1)\|^2_{X\times D(A^{1/2})'} \leqslant \|\chi_Q u\|^2_{L^2((0,T)\times\Omega)} = \int_0^T \int_{\omega(t)} |u(t,x)|^2 \, dx_g \, dt, \tag{6}$$

for all $(u^0, u^1) \in X \times D(A^{1/2})'$, where u is the solution of (3)*-*(4) *with initial conditions $u_{|t=0} = u^0$ and $\partial_t u_{|t=0} = u^1$.*

This proposition is proven in Section 2.2.

In the existing literature, the observation is most often made on cylindrical domains $Q = (0, T) \times \omega$ for some given $T > 0$, meaning that $\omega(t) = \omega$. In such a case where the observation domain ω is stationary, it is known that, within the class of smooth domains Ω, the observability property holds if the pair (ω, T) satisfies the *Geometric Control Condition* (in short, GCC) in Ω (see [2, 4]). Roughly speaking, it says that every geodesic propagating in Ω at unit speed, and reflecting at the boundary according to the classical laws of geometrical optics, so-called generalized geodesics, should meet the open set ω within time T.

In the present article, our goal is to extend the GCC to *time-dependent observation domains*. For a precise statement of the GCC, we first recall the definition of generalized geodesics and bicharacteristics.

1.3.1 The generalized bicharacteristic flow of R. Melrose and J. Sjöstrand

First, we define generalized bicharacteristics in the interior of Ω. There, they coincide with the classical notion of bicharacteristics. Second, we define generalized bicharacteristics in the neighborhood of the boundary.

Symbols and bicharacteristics in the interior. The principal symbol of $-\triangle_g$ coincides with the cometric g^* defined by

$$g^*_x(\xi, \xi) = \max_{v \in T_x M \setminus \{0\}} \frac{\langle \xi, v\rangle^2}{g_x(v, v)},$$

for every $x \in M$ and every $\xi \in T^*_x M$. In local coordinates, we denote by $g_{ij}(x)$ the Riemannian metric g at point x, that is, $g(v, \tilde{v})(x) = g_{ij}(x)v^i(x)\tilde{v}^j(x)$, for $v, \tilde{v} \in TM$, that is, two vector fields, and by $g^{ij}(x)$ the cometric g^* at x, that is $g^*(\omega, \tilde{\omega})(x) = g^{ij}(x)\omega_i(x)\tilde{\omega}_j(x)$, for $\omega, \tilde{\omega} \in T^*M$, that is, two 1-forms. In local coordinates, the Laplace-Beltrami reads

$$-\triangle_g = -\mathsf{g}(x)^{-1/2}\partial_i(\mathsf{g}(x)^{1/2} g^{ij}(x)\partial_j).$$

In $\mathbb{R}\times M$, the principal symbol of the wave operator $\partial_t^2 - \triangle_g$ is then $p(t, x, \tau, \xi) = -\tau^2 + g^*_x(\xi, \xi)$. In $T^*(\mathbb{R} \times M)$, the Hamiltonian vector field H_p associated with p is given by $H_p f = \{p, f\}$, for $f \in \mathscr{C}^1(T^*(\mathbb{R} \times M))$. In local coordinates, H_p reads

$$H_p = \partial_\tau p \, \partial_t + \nabla_\xi p \, \nabla_x - \nabla_x p \, \nabla_x = -2\tau\partial_t + 2g^{jk}(x)\xi_k\partial_{x_j} - \partial_{x_j} g^{ik}(x)\xi_i\xi_k\partial_{\xi_j},$$

with the usual Einstein summation convention. Along the integral curves of H_p, the value of p is constant as $H_p p = 0$. Thus, the characteristic set $\mathrm{Char}(p) = \{p = 0\}$ is invariant under the flow of H_p. In $T^*(\mathbb{R} \times M)$, bicharacteristics are defined as the maximal integral curves of H_p that lay in $\mathrm{Char}(p)$. The projections of the bicharacteristics onto M, using the variable t as a parameter, coincide with the geodesics on M associated with the metric g travelled at speed one.

We set $Y = \mathbb{R} \times \overline{\Omega}$. We denote by $\mathrm{Char}_Y(p)$ the characteristic set of p above Y, given by

$$\mathrm{Char}_Y(p) = \{\rho = (t, x, \tau, \xi) \in T^*(\mathbb{R} \times M) \setminus 0 \ \mid \ x \in \overline{\Omega} \text{ and } p(\rho) = 0\}.$$

Coordinates and Hamiltonian vector fields near and at the boundary. Close to the boundary $\mathbb{R} \times \partial\Omega$, using normal geodesic coordinates (x', x_d), the principal symbol of the Laplace-Beltrami operator reads $\xi_d^2 + \ell(x, \xi')$. Set $y = (t, x)$, $y' = (t, x')$ and $y_n = x_d$. Here $n = d + 1$. In these coordinates, the principal symbol of the wave operator takes the form $p(y', y_n, \eta', \eta_n) = \eta_n^2 + r(y, \eta')$, where r is a smooth y_n-family of tangential (differential) symbols, and the boundary $\mathbb{R} \times \partial\Omega$ is locally parametrized by y' and given by $\{y_n = 0\}$. The open set $\mathbb{R} \times \Omega$ is locally given by $\{y_n > 0\}$.

The variables $\eta = (\eta', \eta_n)$ are the cotangent variables associated with $y = (y', y_n)$. We set

$$\partial T^*Y = \{\rho = (y, \eta) \in T^*(\mathbb{R} \times M) \ \mid \ y_n = 0\},$$

as the boundary of $T^*Y = \{\rho = (y, \eta) \in T^*(\mathbb{R} \times M) \ \mid \ y \in Y\}$. In those local coordinates, the associated Hamiltonian vector field H_p is given by

$$H_p = \nabla_{\eta'} r \nabla_{y'} + 2\eta_n \partial_{y_n} - \nabla_y r \nabla_\eta.$$

We denote by r_0 the trace of r on ∂T^*Y, that is, $r_0(y', \eta') = r(y', y_n = 0, \eta')$. We then introduce the Hamiltonian vector field above the submanifold $\{y_n = 0\}$

$$H_{r_0} = \nabla_{\eta'} r_0 \nabla_{y'} - \nabla_{y'} r_0 \nabla_{\eta'}.$$

The compressed cotangent bundle. On Y, for points $y = (y', y_n)$ near the boundary, we define the vector fiber bundle ${}^bTY = \cup_{y \in Y} {}^bT_yY$, generated by the vector fields $\partial_{y'}$ and $y_n \partial_{y_n}$, in the local coordinates introduced above. We then have the natural map

$$\begin{aligned} j : T^*Y &\longrightarrow {}^bT^*Y = \bigcup_{y \in Y} ({}^bT_yY)^*, \\ (y; \eta) &\longmapsto (y; \eta', y_n \eta_n), \end{aligned}$$

expressed here in local coordinates for simplicity. In particular:

- If $y \in \mathbb{R} \times \Omega$ then ${}^bT_y^*Y = j(T_y^*Y)$ is isomorphic to $T_y^*Y = T_y^*(\mathbb{R} \times M)$;
- if $y \in \mathbb{R} \times \partial\Omega$ then ${}^bT_y^*Y = j(T_y^*Y)$ is isomorphic to $T_y^*(\mathbb{R} \times \partial\Omega)$.

The bundle ${}^bT^*Y$ is called the *compressed cotangent bundle*, and we see that it allows one to patch together $T_y^*(\mathbb{R} \times M)$ in the interior of Ω and $T_y^*(\mathbb{R} \times \partial\Omega)$ at the boundary in a smooth manner, despite the discrepancy in their dimensions.

Decomposition of the characteristic set at the boundary. We set $\Sigma = j(\mathrm{Char}_Y(p)) \subset {}^bT^*Y$ and $\Sigma_0 = \Sigma_{|y_n=0} \subset \partial\, {}^bT^*Y = {}^bT^*Y_{|y_n=0} \simeq T^*(\mathbb{R} \times \partial\Omega)$. Using local coordinates (for convenience here), we then define $G \subset \Sigma_0$ by $r(y, \eta') = r_0(y', \eta') = 0$ as the glancing set and $H = \Sigma_0 \setminus G$ as the hyperbolic set. Hence, if $\rho = (y', y_n = 0, \eta') \in \Sigma_0$ then

$$\rho \in H \ \Leftrightarrow \ r_0(y', \eta') < 0, \qquad \rho \in G \ \Leftrightarrow \ r_0(y', \eta') = 0.$$

The set of points $(y', y_n = 0, \eta') \in {}^bT^*Y_{|y_n=0}$ such that $r_0(y', \eta') > 0$ is referred to as the elliptic set E. We also set $\hat{\Sigma} = \Sigma \cup E = \Sigma \cup {}^bT^*Y_{|y_n=0}$ and the following cosphere quotient space $S^*\hat{\Sigma} = \hat{\Sigma}/(0, +\infty)$.

The glancing set is itself written as $G = G^2 \supset G^3 \supset \cdots \supset G^\infty$, with $\rho = (y', y_n = 0, \eta)$ in G^{k+2} if and only if

$$\eta_n = r_0(\rho) = 0, \quad H_{r_0}^j r_1(\rho) = 0, \quad 0 \leqslant j < k,$$

where $r_1(\rho) = \partial_{y_n} r(y', y_n = 0, \eta')$. Finally, we write $G^2 \setminus G^3$, the set of glancing points of order exactly 2, as the union of the diffractive set G_d^2 and of the gliding set G_g^2, that is $G^2 \setminus G^3 = G_d^2 \cup G_g^2$, with

$$\rho \in G_d^2 \text{ (resp., } G_g^2) \ \Leftrightarrow\ \rho \in G^2 \setminus G^3 \text{ and } r_1(\rho) < 0 \text{ (resp., } > 0).$$

Similarly, for $\ell \geqslant 2$, we write $G^{2\ell} \setminus G^{2\ell+1}$, the set of glancing points of order exactly $k = 2\ell$, as the union of the diffractive set $G_d^{2\ell}$ and the gliding set $G_g^{2\ell}$, that is $G^{2\ell} \setminus G^{2\ell+1} = G_d^{2\ell} \cup G_g^{2\ell}$, with

$$\rho \in G_d^{2\ell} \text{ (resp. } G_g^{2\ell}) \ \Leftrightarrow\ \rho \in G^{2\ell} \setminus G^{2\ell+1} \text{ and } H_{r_0}^{2\ell-2} r_1(\rho) < 0 \text{ (resp. } > 0).$$

We shall call diffractive[2] a point in $G_d = \cup_{\ell \geqslant 1} G_d^{2\ell}$.

Observe that a bicharacteristic going through a point of G^k projects onto a geodesic on M that has a contact of order k with $\mathbb{R} \times \partial\Omega$.

Generalized bicharacteristics. What we introduced above now allows us to give a precise definition of generalized bicharacteristics above Y.

Definition 1.3. A generalized bicharacteristic of p is a differentiable map

$$\mathbb{R} \setminus B \ni s \mapsto \gamma(s) \in (\mathrm{Char}_Y(p) \setminus \partial T^*Y) \cup j^{-1}(G),$$

where B is a subset of $\mathbb{R}$ made of isolated points, that satisfies the following properties

(i) $\gamma'(s) = H_p(\gamma(s))$ if either $\gamma(s) \in \mathrm{Char}_Y(p) \setminus \partial T^*Y$ or $\gamma(s) \in j^{-1}(G_d)$;

(ii) $\gamma'(s) = H_{r_0}(\gamma(s))$ if $\gamma(s) \in j^{-1}(G \setminus G_d)$;

(iii) If $s_0 \in B$, there exists $\delta > 0$ such that $\gamma(s) \in \mathrm{Char}_Y(p) \setminus \partial T^*Y$ for $s \in (s_0 - \delta, s_0) \cup (s_0, s_0 + \delta)$. Moreover, the limits $\rho^\pm = (y^\pm, \eta^\pm) = \lim_{s \to s_0^\pm} \gamma(s)$ exist and $y_n^- = y_n^+ = 0$, i.e., $\rho^\pm \in \mathrm{Char}_Y(p) \cap \partial T^*Y$, $y^{-\prime} = y^{+\prime}$, $\eta^{-\prime} = \eta^{+\prime}$, and $\eta_n^- = -\eta_n^+$. That is, ρ^+ and ρ^- lay in the same hyperbolic fiber above a point in ${}^bT^*Y_{|y_n=0}$: $j(\rho^+) = j(\rho^-) \in H$.

Point (i) describes the generalized bicharacteristic in the interior, that is, in $T^*(\mathbb{R} \times \Omega)$, and at diffractive points, where it coincides with part of a classical bicharacteristic as defined above. Point (ii) describes the behavior in $G \setminus G_d$, thus explaining that a generalized bicharacteristic can enter or leave the boundary ∂T^*Y or locally remain in it. Point (iii) describes reflections when the boundary ∂T^*Y is reached transversally by a classical bicharacteristic, that is, at a point of the hyperbolic set. While $s \mapsto \xi(s)$ exhibits a jump at such a point, $s \mapsto t(s)$ and $s \mapsto x(s)$ can both be extended by continuity there. We shall thus proceed with this extension. Above, for clarity we chose to state Point (iii) in local coordinates near the hyperbolic point. The generalized bicharacteristics are however defined as a geometrical object, independent of the choice of coordinates.

[2] In the sense of Taylor and Melrose the terminology diffractive only applies to G_d^2. Here, we chose to extend it to $\cup_{\ell \geqslant 1} G_d^{2\ell}$ as we shall refer to nondiffrative points, that is, the complement of G_d, in Section 4. In the literature nondiffractive points are defined this way but diffractive points are often defined according to Taylor and Melrose. Then, the set of nondiffractive points and the set of diffractive points are not complements of one another; a source of confusion.

Definition 1.4. Compressed generalized bicharacteristics are the image under the map j of the generalized bicharacteristics defined above.

If ${}^b\gamma = j(\gamma)$ is such a compressed generalized bicharacteristics, then ${}^b\gamma : \mathbb{R} \to {}^bT^*Y \setminus E$ is a continuous map (if one introduces the proper natural topology on ${}^bT^*Y$).

Using t as parameter, generalized geodesics for Ω, travelled at speed one, are then the projection on M of the (compressed) generalized bicharacteristics. Generalized geodesics remain in $\overline{\Omega}$. We shall call a "ray" this projection following the terminology of geometrical optics.

An important result is then the following.

Proposition 1.5. *A (compressed) generalized bicharacteristic with no point in G^∞ is uniquely determined by any one of its points.*

We refer to [15] for a proof of this result and for many more details on generalized bicharacteristics (see also [10, Section 24.3]).

1.3.2 A time-dependent geometric control condition

With the notion of compressed generalized bicharacteristic recalled in Section 1.3.1, we can state the geometric condition adapted to a time-dependent control domain.

Definition 1.6. Let Q be an open subset of $\mathbb{R} \times \overline{\Omega}$, and let $T > 0$. We say that (Q, T) satisfies the *time-dependent Geometric Control Condition* (in short, t-GCC) if every generalized bicharacteristic ${}^b\gamma : \mathbb{R} \ni s \mapsto (t(s), x(s), \tau(s), \xi(s)) \in {}^bT^*Y \setminus E$ is such that there exists $s \in \mathbb{R}$ such that $t(s) \in (0, T)$ and $(t(s), x(s)) \in Q$. We say that Q satisfies the t-GCC if there exists $T > 0$ such that (Q, T) satisfies the t-GCC.

The control time $T_0(Q, \Omega)$ is defined by

$$T_0(Q, \Omega) = \inf\{T > 0 \ \mid \ (Q, T) \text{ satisfies the } t\text{-GCC}\},$$

with the agreement that $T_0(Q, \Omega) = +\infty$ if Q does not satisfy the t-GCC.

The t-GCC property of Definition 1.6 is a time-dependent version of the usual GCC.

Remark 1.7. Several remarks are in order.

1. The t-GCC assumption implies that the set $\mathcal{O} = \cup_{t \in (0,T)} \omega(t)$ is a control domain that satisfies the usual GCC for a time $T > T_0(Q, \Omega)$.

2. It is interesting to note that the control time $T_0(Q, \Omega)$ is not a continuous function of the domains, for any reasonable topology (see Remark 3.1 below).

3. Observe that if (Q, T) satisfies the t-GCC, a similar geometric condition may not occur if the time interval $(0, T)$ is replaced by $(t_0, t_0 + T)$. As the set Q is not a cylinder in general, by nature the t-GCC is not invariant under time translation.

4. Note that Q cannot be chosen as an open set of $\mathbb{R} \times \Omega$ instead of $\mathbb{R} \times \overline{\Omega}$. Consider indeed the case of a disk, if Q is an open set of $\mathbb{R} \times \Omega$ then the ray that glides along the boundary never enters Q. This coincides with the so-called whispering gallery phenomenon.

1.4 Main result

Theorem 1.8. *Let Q be an open subset of $\mathbb{R} \times \overline{\Omega}$ that satisfies the t-GCC. Let $T > T_0(Q, \Omega)$. If $\partial\Omega \neq \emptyset$, we assume moreover that no generalized bicharacteristic has a contact of infinite order with $(0, T) \times \partial\Omega$, that is, $G^\infty = \emptyset$. Then, the observability inequality (5) holds.*

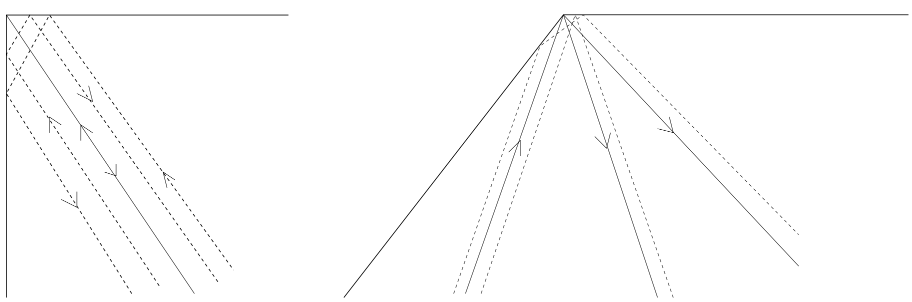
Theorem 1.8 is proven in Section 2.1. By Proposition 1.2 we have the following result.

Theorem 1.8′. *Under the same assumptions as Theorem 1.8 the observability inequality (6) holds.*

Remark 1.9.

1. In the case where $\partial\Omega \neq \emptyset$, the assumption of the absence of any ray having a contact of infinite order with the boundary is classical (see [2]). Note that this assumption is not useful if M and $\partial\Omega$ are analytic. This assumption is used in a crucial way in the proof of the theorem to ensure uniqueness of the generalized bicharacteristic flow, as stated in Proposition 1.5.

2. We have assumed here that, if $\partial\Omega \neq \emptyset$, then $\partial\Omega$ is smooth. The case where $\partial\Omega$ is not smooth is open. Even the case where $\partial\Omega$ is piecewise analytic is open. The problem is that, in that case, the generalized bicharacteristic flow is not well defined since there is no uniqueness of a bicharacteristic passing over a point. This fact is illustrated on Figure 1(b) where a ray reflecting at some angle can split into two rays. However, it clearly follows from our proof that Theorem 1.8 is still valid if the domain Ω is such that this uniqueness property holds (like in the case of a rectangle). In general, we conjecture that the conclusion of Theorem 1.8 holds true if *all* generalized bicharacteristics meet T^*Q within time T. This would require however to extend the classical theory of propagation of singularities. Proving this fact is beyond of the scope of the present article. We may however assert here, in the present context, that the result of Theorem 1.8 is valid in any d-dimensional orthotope.



(a) Right angle corner.

(b) General corner.

Figure 1: Reflection of generalized geodesics at some corner.

Remark 1.10. In the case of a 1D wave equation with Dirichlet boundary conditions, the corresponding statement of Theorem 1.8 is proven in [7] by means of the D'Alembert formula. The proof we provide in Section 2.1, is general and follows [2, 4]. In fact, as already mentioned in Section 1.1, a key step of the approach of [2, 4] is simplified here. More precisely, the approach of [2, 4] consists of several steps. Firstly, a weaker version of the observability inequality is proven; in the present article, this is done in Lemma 2.1. Secondly, the so-called *set of invisible solutions* (defined by (16)) is shown to be reduced to zero; in the present article, this is done in Lemma 2.3.

Thirdly, the observability inequality is proven to hold by means of the result of the two previous steps. Our simplification with respect to [2, 4] lies in the second step. The argument is much shorter than the original one and, in the present analysis of a time varying observation region, it turns out to be crucial, as the more classical argument of [2, 4] cannot be applied.

1.5 Consequences

1.5.1 Controllability

By the usual duality argument (Hilbert Uniqueness Method, see [14, 13]), we have the following equivalent result for the control of the wave equation with a time-dependent control domain, based on the observability inequality (6) that follows from Theorem 1.8′.

Theorem 1.8″. *Let Q be an open subset of $\mathbb{R}\times\overline{\Omega}$ that satisfies the t-GCC. Let $T > T_0(Q,\Omega)$. If $\partial\Omega \neq \emptyset$ we assume moreover that no generalized bicharacteristic has a contact of infinite order with $(0,T)\times\partial\Omega$, that is, $G^\infty = \emptyset$. Setting $\omega(t) = \{x \in \Omega \mid (t,x) \in Q\}$, we consider the wave equation with internal control*

$$\partial_t^2 u - \triangle_g u = \chi_Q f, \tag{7}$$

in $(0,T)\times\Omega$, with Dirichlet or Neumann boundary conditions (4) whenever $\partial\Omega \neq \emptyset$, and with $f \in L^2((0,T)\times\Omega)$. Then, the controlled equation (7) is exactly controllable in the space $D(A^{1/2})\times X$, meaning that, for all (u^0,u^1) and (v^0,v^1) in $D(A^{1/2})\times X$, there exists $f \in L^2((0,T)\times\Omega)$ such that the corresponding solution of (7), with $(u_{|t=0}, \partial_t u_{|t=0}) = (u^0,u^1)$, satisfies $(u_{|t=T}, \partial_t u_{|t=T}) = (v^0,v^1)$.

Remark 1.11. In the above result the control operator is $f \mapsto \chi_Q f$. We could choose instead a control operator $f \mapsto b(t,x)f$ with b smooth and such that $Q = \{(t,x) \in \mathbb{R}\times\overline{\Omega} \mid b(t,x) > 0\}$. Then with the same t-GCC we also have exact controllability in this case. The equivalent observability inequality is then

$$C\|(u^0,u^1)\|^2_{X\times D(A^{1/2})'} \leqslant \|bu\|^2_{L^2((0,T)\times\Omega)} = \int_0^T\int_\Omega |b(t,x)u(t,x)|^2\,dx_g\,dt.$$

1.5.2 Observability with few sensors

We give another interesting consequence of Theorem 1.8, in connection with the very definition of the t-GCC property, which can be particularly relevant in view of practical applications.

Corollary 1.12. *Let $Q \subset \mathbb{R}\times\overline{\Omega}$ be an open subset with Lipschitz boundary and let $T > 0$ be such that (Q,T) satisfies the t-GCC. Then, every open subset $\mathcal{V}$ of $[0,T]\times\overline{\Omega}$ (for the topology induced by $\mathbb{R}\times M$), containing $\partial\left(Q\cap([0,T]\times\overline{\Omega})\right)$, is such that $(\mathcal{V},T)$ satisfies the t-GCC and, consequently, observability holds for such an open subset.*

Proof. Let $\mathbb{R}\ni s \mapsto {}^b\gamma(s)$ be a compressed generalized bicharacteristic with $t = t(s)$. As (Q,T) satisfies the t-GCC, there exists $t_1 \in (0,T)$ and $s_1 \in \mathbb{R}$ such that $t_1 = t(s_1)$ and ${}^b\gamma(s_1) \in j(T^*(Q))$. Now, there are two cases.

Case 1. There exists $s_2 \in \mathbb{R}$ such that $t_2 = t(s_2) \in (0,T)$ and ${}^b\gamma(s_2) \notin j(T^*(Q))$. The continuity of $s \mapsto {}^b\gamma(s)$ into ${}^bT^*Y \setminus E$, in particular of its projection on $\mathbb{R}\times\overline{\Omega}$, then allows one to conclude that there exists $s_3 \in \mathbb{R}$ such that $t_3 = t(s_3) \in (0,T)$ and ${}^b\gamma(s_3) \in j(T^*(\mathcal{V}))$.

Case 2. For all $s \in \mathbb{R}$ such that $t = t(s) \in (0,T)$ we have ${}^b\gamma(s) \in j(T^*(Q))$. Such a bicharacteristic is illustrated in Figures 2(b) and 2(c). Then $s \mapsto {}^b\gamma(s)$ enters $j(T^*(\mathcal{W}))$ for any neighborhood $\mathcal{W}$ of $\{T\}\times\omega(T)$ (or $\{0\}\times\omega(0)$). Thus, there exists $s_2 \in \mathbb{R}$ such that $t_2 = t(s_2) \in (0,T)$ and ${}^b\gamma(s_2) \in j(T^*(\mathcal{V}))$.

$\square$

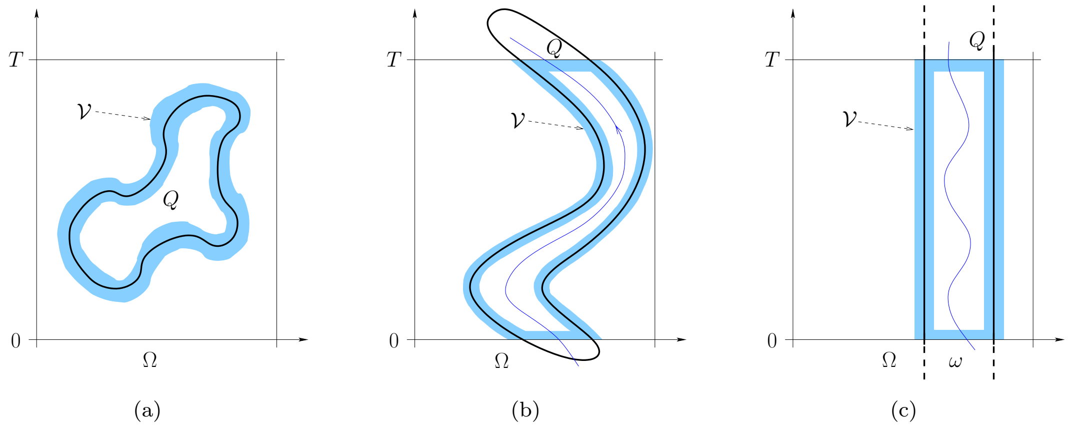


Figure 2: Neighborhood $\mathcal{V}$ of $\partial\left(Q \cap ([0,T] \times \overline{\Omega})\right)$ in $[0,T] \times \overline{\Omega}$ (for the induced topology). In 2(b) and 2(c), a potential bicharacteristic that remains in the interior of Q is represented.

Remark 1.13.

1. The main interest of Corollary 1.12 is that it allows one to take the open set $\mathcal{V}$ "as small as possible", provided that it contains the boundary of $Q \cap ([0,T] \times \overline{\Omega})$ (see Figure 2). As a practical consequence, only few sensors are needed to ensure the observability property, or, by duality, the controllability property, thus reducing the cost of an experiment.

   Somehow, with an internal control we have $d+1$ degrees of freedom for the control of d variables, and this explains intuitively why the choice of a "thin" open set $\mathcal{V}$ is possible. The above corollary roughly states that control is still feasible by using only d degrees of freedom. In terms of the control domain, this means that we only need a control domain that is any open neighborhood of a set of Hausdorff dimension d.

2. Observe that the proof of Corollary 1.12 and Figures 2(b) and 2(c) show in fact that it suffices to choose $\mathcal{V}$ as the union of a neighborhood of $\partial Q \cap (0,T) \times \overline{\Omega}$ and a neighborhood of $Q \cap \{t = 0\}$ (or $Q \cap \{t = T\}$).

3. Note that, if an open subset ω of $\overline{\Omega}$ satisfies the usual GCC, then a small neighborhood of $\partial\omega$ does not satisfy necessarily the GCC. In contrast, when considering time-space control domains (i.e., subsets of $\mathbb{R} \times \overline{\Omega}$), the situation is different. For instance, if ω satisfies the GCC, then any open subset of $[0,T] \times \overline{\Omega}$, containing $\partial([0,T] \times \omega) = ([0,T] \times \partial\omega) \cup (\{0\} \times \omega) \cup (\{T\} \times \omega)$, satisfies the t-GCC (see Figure 2(c)).

4. Note that $T(Q,\Omega) \leqslant \liminf T(\mathcal{V},\Omega)$ as $\mathcal{V}$ shrinks to $\partial\left(Q \cap ([0,T] \times \overline{\Omega})\right)$, and that equality may fail as there may exist some bicharacteristics propagating inside Q and not reaching $\mathcal{V}$ for $t \in (t_1, t_2)$ for with $0 < t_1 < t_2 < T$ (see Figures 2(b) and 2(c)).

1.5.3 Stabilization

Theorem 1.8 has the following consequence for wave equations with a damping localized on a domain Q that is time-periodic.

Corollary 1.14. *Let Q be an open subset of $\mathbb{R} \times \overline{\Omega}$, satisfying the t-GCC. Let $T > T_0(Q, \Omega)$. If $\partial\Omega \neq \emptyset$, we assume moreover that no generalized bicharacteristic has a contact of infinite order with $(0,T) \times \partial\Omega$, that is, $G^\infty = \emptyset$. Setting $\omega(t) = \{x \in \Omega \mid (t,x) \in Q\}$, we assume that ω is T-periodic, that is, $\omega(t+T) = \omega(t)$, for almost every $t \in (0,T)$. We consider the wave equation with a local internal damping term,*

$$\partial_t^2 u - \triangle_g u + \chi_\omega \partial_t u = 0, \tag{8}$$

in $(0,T) \times \Omega$, with Dirichlet or Neumann boundary conditions (4) whenever $\partial\Omega \neq \emptyset$. Then, there exists $\mu \geqslant 0$ and $\nu > 0$ such that any solution of (7), with $(u(0), \partial_t u(0)) \in D(A^{1/2}) \times X$, satisfies

$$E_0(u)(t) \leqslant \mu E_0(u)(0) e^{-\nu t},$$

where we have set $E_0(u)(t) = \frac{1}{2}\big(\|A^{1/2}u(t)\|^2_{L^2(\Omega)} + \|\partial_t u(t)\|^2_{L^2(\Omega)}\big)$.

Corollary 1.14 is proven in Section 2.3.

2 Proofs

2.1 Proof of Theorem 1.8

The proof follows the classical chain of arguments developed in [2, 4], with yet a simplification of one of the key steps, as already pointed out in Remark 1.10. This simplification is a key element here. The original proof scheme would not allow one to conclude in the case of a time-dependent control domain.

For a solution u of (3)-(4), with $u_{|t=0} = u^0 \in D(A^{1/2})$ and $\partial_t u_{|t=0} = u^1 \in X$, we use the natural energy

$$E_0(u)(t) = \frac{1}{2}\big(\|u(t)\|^2_{D(A^{1/2})} + \|\partial_t u(t)\|^2_X\big). \tag{9}$$

In the proof, we use the fact that this energy remains constant as time evolves, that is,

$$E_0(u)(t) = E_0(u)(0) = \frac{1}{2}\big(\|u^0\|^2_{D(A^{1/2})} + \|u^1\|^2_X\big) = \frac{1}{2}\|(u^0, u^1)\|^2_{D(A^{1/2}) \times X}, \tag{10}$$

and we shall simply write $E_0(u)$ at places.

We first achieve a weak version of the observability inequality.

Lemma 2.1. *There exists $C > 0$ such that*

$$C\|(u^0, u^1)\|^2_{D(A^{1/2}) \times X} \leqslant \|\chi_Q \partial_t u\|^2_{L^2((0,T)\times\Omega)} + \|(u^0, u^1)\|^2_{X \times D(A^{1/2})'}, \tag{11}$$

for all $(u^0, u^1) \in D(A^{1/2}) \times X$, where u is the corresponding solution of (3)-(4) with $u_{|t=0} = u^0$ and $\partial_t u_{|t=0} = u^1$.

Remark 2.2. With respect to the desired observability inequality (5), the inequality (11) exhibits a penalization term, $\|(u^0, u^1)\|_{X \times D(A^{1/2})'}$, on the right-hand side. Note that the Sobolev spaces under consideration have no importance, and instead of $X \times D(A^{1/2})'$ we could as well have chosen $H^{1/2-s}(\Omega) \times H^{-1/2-s}(\Omega)$, for any $s > 0$. The key point lies in the fact that the space $D(A^{1/2}) \times X$ is compactly embedded into $X \times D(A^{1/2})'$.

Proof. We prove the result by contradiction. We assume that there exists a sequence $(u_n^0, u_n^1)_{n\in\mathbb{N}}$ in $D(A^{1/2}) \times X$ such that

$$\|(u_n^0, u_n^1)\|_{D(A^{1/2})\times X} = 1 \quad \forall n \in \mathbb{N}, \tag{12}$$

$$\|(u_n^0, u_n^1)\|_{X\times D(A^{1/2})'} \xrightarrow[n\to+\infty]{} 0, \tag{13}$$

and

$$\|\chi_Q \partial_t u_n\|_{L^2((0,T)\times\Omega)} \xrightarrow[n\to+\infty]{} 0, \tag{14}$$

where u_n is the solution of (3)-(4) satisfying $u_{n|t=0} = u_n^0$ and $\partial_t u_{n|t=0} = u_n^1$. From (12), the sequence $(u_n^0, u_n^1)_{n\in\mathbb{N}}$ is bounded in $D(A^{1/2}) \times X$, and using (13) the only possible closure point for the weak topology of $D(A^{1/2}) \times X$ is $(0, 0)$. Therefore, the sequence $(u_n^0, u_n^1)_{n\in\mathbb{N}}$ converges to $(0, 0)$ for the weak topology of $D(A^{1/2}) \times X$. By continuity of the flow with respect to initial data, it follows that the sequence $(u_n)_{n\in\mathbb{N}}$ of corresponding solutions converges to 0 for the weak topology of $H^1((0,T) \times \Omega)$; in particular, it is bounded.

Up to a subsequence (still denoted $(u_n)_{n\in\mathbb{N}}$ in what follows), according to Proposition A.1 in Appendix A, there exists a microlocal defect measure μ on the cosphere quotient space $S^*\hat{\Sigma}$ introduced in Section 1.3.1, such that

$$(Ru_n, u_n) \xrightarrow[n\to+\infty]{} \langle \mu, \kappa(R)\rangle, \tag{15}$$

for every $R \in \Psi^0(Y)$ with $\kappa(R)$ to be understood as a continuous function on $S^*\hat{\Sigma}$.

It follows from (14) that μ vanishes in $j(T^*Q) \cap S^*\hat{\Sigma}$. As is well known, the measure μ is invariant[3] under the compressed generalized bicharacteristic flow [11, 6]. The definition of this flow is recalled in Section 1.3.1.

The t-GCC assumption for Q then implies that μ vanishes identically (see [2, 4]). This precisely means that $(u_n)_{n\in\mathbb{N}}$ strongly converges to 0 in $H^1((0,T) \times \Omega)$.

Now, we let $0 < t_1 < t_2 < T$. The above strong convergence implies that

$$\int_{t_1}^{t_2} E_0(u_n)(t)\,dt \xrightarrow[n\to+\infty]{} 0,$$

with the energy E_0 defined in (9). As this energy is preserved (with respect to time t), this implies that

$$E_0(u_n)(0) = \frac{1}{2}\big(\|u_n^0\|^2_{D(A^{1/2})} + \|u_n^1\|^2_X\big) \xrightarrow[n\to+\infty]{} 0,$$

yielding a contradiction. $\square$

We define the set of *invisible solutions* as

$$N_T = \{v \in H^1((0,T)\times\Omega) \ \mid \ v \text{ is a solution of (3)-(4)}, \\ \text{with } v_{|t=0} \in D(A^{1/2}),\ \partial_t v_{|t=0} \in X \ \text{ and } \chi_Q \partial_t v = 0\}, \tag{16}$$

equipped with the norm $\|v\|^2_{N_T} = \|v_{|t=0}\|^2_{D(A^{1/2})} + \|\partial_t v_{|t=0}\|^2_X$. Clearly, N_T is closed.

[3] The theorem of propagation for measures is proven in [11] for a damped wave equation with Dirichlet boundary condition. We claim that the same proof applies with Neumann boundary condition. First, using the notations in [11], we still have $\mu_\partial = 0$ in the Neumann case, where μ_∂ is defined by (A.18) in [11]. Then, the proof of theorem A.1 in [11] about the propagation of the measure still applies in the Neumann case. First, one can assume that the tangential operator Q_0 that appears in (A.28) satisfies $\partial_x Q_0|_{x=0} = 0$, to assure that $Q_0 u$ still satisfies the Neumann boundary condition. Then inequality (A.29) in [11] holds true as well, since the theorem of propagation of Melrose-Sjöstrand holds true in the Neumann case [15, 16]. Finally, estimate (A.33) of [11] remains valid since the energy estimate holds true in the Neumann case.

Lemma 2.3. *We have $N_T = \{0\}$.*

In other words, due the t-GCC assumption, there is no nontrivial invisible solution.

Proof. First, the t-GCC assumption combined with the propagation of singularities along the generalized bicharacteristic flow (see [10, Theorem 24.5.3]) implies that all elements of N_T are smooth functions on $(0,T)\times\Omega$, up to the boundary. In particular, if $v\in N_T$ then $\partial_t v\in N_T$.

Second, we remark that, since the operator $\partial_t^2 - \triangle_g$ is time-independent (as well as the boundary condition), the space N_T is invariant under the action of the operator ∂_t.

Third, applying the weak observability inequality of Lemma 2.1 gives

$$C\|v\|_{N_T}^2 = C\|(v_{|t=0}, \partial_t v_{|t=0})\|_{D(A^{1/2})\times X} \leqslant \|(v_{|t=0}, \partial_t v_{|t=0})\|_{X\times D(A^{1/2})'},$$

for every $v\in N_T$. Since $D(A^{1/2})\times (D(A^{1/2}))'$ is compactly embedded into $X\times D(A^{1/2})'$, this implies that the unit ball of N_T is compact and thus N_T is finite dimensional.

We are now in a position to prove the lemma. The proof goes by contradiction. Let us assume that $N_T\neq\{0\}$. The operator $\partial_t : N_T\to N_T$ has at least one (complex) eigenvalue λ, associated with an eigenfunction $v\in N_T\setminus\{0\}$. Since $\partial_t v=\lambda v$, it follows that $v(t,x)=e^{\lambda t}w(x)$, and since $(\partial_t^2-\triangle_g)v=0$ we obtain $(\lambda^2-\triangle_g)w=0$. Note that $\lambda\neq 0$ (in the Neumann case, we have $w\in L^2_0(\Omega)$). Now, take any $t\in(0,T)$ such that $\omega(t)=\{x\in\Omega\ \mid\ (t,x)\in Q\}\neq\emptyset$. Since $\chi_Q\partial_t v=0$ and thus $\chi_Q v=0$, it follows that $w=0$ on the open set $\omega(t)$. By elliptic unique continuation we then infer that $w=0$ on the whole Ω, and hence $v=0$. This is a contradiction. □

Let us finally derive the observability inequality (5). To this aim, the compact term on the right-hand-side of (11) must be removed. We argue again by contradiction, assuming that there exists a sequence $(u_n^0,u_n^1)_{n\in\mathbb{N}}$ in $D(A^{1/2})\times X$ such that

$$\|(u_n^0,u_n^1)\|_{D(A^{1/2})\times X}=1,\quad \forall n\in\mathbb{N}, \tag{17}$$

$$\|\chi_Q\partial_t u_n\|_{L^2((0,T)\times\Omega)} \xrightarrow[n\to+\infty]{} 0, \tag{18}$$

where u_n is the solution of (3)-(4) such that $u_{n|t=0}=u_n^0$ and $\partial_t u_{n|t=0}=u_n^1$. From (17), the sequence $(u_n^0,u_n^1)_{n\in\mathbb{N}}$ is bounded in $D(A^{1/2})\times X$, and therefore, extracting if necessary a subsequence, it converges to some $(u^0,u^1)\in D(A^{1/2})\times X$ for the weak topology. Let u be the solution of (3)-(4) such that $u_{|t=0}=u^0$ and $\partial_t u_{|t=0}=u^1$. Then, $\chi_Q\partial_t u_n\to\chi_Q\partial_t u$ weakly in $L^2((0,T)\times\Omega)$ implying

$$\|\chi_Q\partial_t u\|_{L^2((0,T)\times\Omega)} \leqslant \liminf_{n\to+\infty}\|\chi_Q\partial_t u_n\|_{L^2((0,T)\times\Omega)}=0,$$

and hence $u\in N_T$. It follows from Lemma 2.3 that $u=0$. In particular, we have then $(u^0,u^1)=(0,0)$ and hence $(u_n^0,u_n^1)_{n\in\mathbb{N}}$ converges to $(0,0)$ for the weak topology of $D(A^{1/2})\times X$, and thus, by compact embedding, for the strong topology of $X\times D(A^{1/2})'$. Applying the weak observability inequality (11) raises a contradiction. This concludes the proof of Theorem 1.8. □

2.2 Proof of Proposition 1.2

First, we assume that the observability inequality (5) holds. Let v be a solution of (3)-(4), with initial conditions $(v^0,v^1)\in X\times D(A^{1/2})'$. We set $u=\int_0^t v(s)ds - A^{-1}v^1$. Then $\partial_t u=v$ and we have $u_{|t=0}=u^0=-A^{-1}v^1\in D(A^{1/2})$, and $\partial_t u_{|t=0}=u^1=v^0\in X$. Moreover, we have

$$\partial_t^2 u(t)=\partial_t v(t)=\int_0^t \partial_t^2 v(s)ds+\partial_t v(0) = -\int_0^t Av(s)ds + v^1 = -Au(t).$$

Since $v = \partial_t u$ and $\|(u^0, u^1)\|_{D(A^{1/2})\times X} = \|(A^{-1}v^1, v^0)\|_{D(A^{1/2})\times X} = \|(v^0, v^1)\|_{X\times D(A^{1/2})'}$, applying the observability inequality (5) to u, we obtain (6).

Second, we assume that the observability inequality (6) holds. Let u be a solution of (3)-(4), with initial conditions $(u^0, u^1) \in D(A^{1/2}) \times X$. We set $v = \partial_t u$. Then v is clearly a solution of (3)-(4), with $v_{|t=0} = v^0 = u^1 \in X$ and $\partial_t v_{|t=0} = v^1 = \partial_t^2 u_{|t=0} = -Au_{|t=0} = -Au^0 \in D(A^{1/2})'$. Since $\|(v^0, v^1)\|_{X\times D(A^{1/2})'} = \|(u^1, Au^0)\|_{X\times D(A^{1/2})'} = \|(u^0, u^1)\|_{D(A^{1/2})\times X}$, applying the observability inequality (6) to $v = \partial_t u$, we obtain (5). □

2.3 Proof of Corollary 1.14

It is proven in [9] that a second-order linear equation with (bounded) damping has the exponential energy decay property, if and only if the corresponding conservative linear equation is observable. The extension to our framework is straightforward. We however give a proof of Corollary 1.14 for completeness.

By Theorem 1.8, there exists $C_0 > 0$ such that

$$C_0\Big(\|A^{1/2}\phi_{|t=0}\|^2_{L^2(\Omega)} + \|\partial_t\phi_{|t=0}\|^2_{L^2(\Omega)}\Big) \leqslant \int_0^S \|\partial_t\phi\|^2_{L^2(\omega(t))}\,dt, \qquad S = \ell T,\ \ell \in \mathbb{N}^*, \tag{19}$$

for ϕ solution of $\partial_t^2\phi = \triangle_g\phi$, with $(\phi_{|t=0}, \partial_t\phi_{|t=0}) \in D(A^{1/2}) \times X$.

Let now u be a solution of (8) with $(u_{|t=0}, \partial_t u_{|t=0}) \in D(A^{1/2}) \times X$. Let us prove that we have an exponential decay for its energy. We consider ϕ as above with the following initial conditions $\phi_{|t=0} = u_{|t=0}$ and $\partial_t\phi_{|t=0} = \partial_t u_{|t=0}$. Then, setting $\theta = u - \phi$, we have

$$\partial_t^2\theta - \triangle_g\theta = \chi_\omega\partial_t u, \qquad \theta_{|t=0} = 0,\ \partial_t\theta_{|t=0} = 0. \tag{20}$$

Observe that $\partial_t u \in L^2(\mathbb{R}\times\Omega)$ yielding $\theta \in \mathscr{C}^0(\mathbb{R}; D(A^{1/2})) \cap \mathscr{C}^1(\mathbb{R}; X)$.

Were the r.h.s. of (20) to be replaced by f in $H^1(\mathbb{R}\times\Omega)$, we would have $\theta \in \mathscr{C}^0(\mathbb{R}; D(A)) \cap \mathscr{C}^1(\mathbb{R}; D(A^{1/2}))$. Recalling the definition of E_0 in the statement of Corollary 1.14, we would find

$$\frac{d}{dt}E_0(\theta)(t) = \langle\partial_t\theta(t), A\theta(t) + \partial_t^2\theta(t)\rangle_{L^2(\Omega)} = \langle\partial_t\theta(t), f\rangle_{L^2(\Omega)}.$$

Continuity with respect to f and a density argument then yield $\frac{d}{dt}E_0(\theta)(t) = \langle\partial_t\theta(t), \chi_\omega\partial_t u\rangle_{L^2(\Omega)}$. With two integrations with respect to $t \in (0, S)$, using that $E_0(\theta)(0) = 0$, we obtain, by the Fubini theorem, $\int_0^S E_0(\theta)(t)\,dt = \int_0^S (S-t)\int_{\omega(t)}\partial_t\theta(t,x)\partial_t u(t,x)\,dx_g\,dt$. With the Young inequality, we have $\int_0^S E_0(\theta)(t)\,dt \leqslant S^2\int_0^S\|\partial_t u\|^2_{L^2(\omega(t))}\,dt + \frac{1}{4}\int_0^S\|\partial_t\theta\|^2_{L^2(\Omega)}\,dt$. With the definition of $E_0(\theta)(t)$, we then infer that

$$\int_0^S \|\partial_t\theta\|^2_{L^2(\Omega)}\,dt \leqslant 4S^2\int_0^S\|\partial_t u\|^2_{L^2(\omega(t))}\,dt. \tag{21}$$

Now, since $\phi = u - \theta$, we have $\|\partial_t\phi\|^2_{L^2(\omega(t))} \leqslant 2\|\partial_t u\|^2_{L^2(\omega(t))} + 2\|\partial_t\theta\|^2_{L^2(\Omega)}$, yielding, using (21),

$$\int_0^S \|\partial_t\phi\|^2_{L^2(\omega(t))}\,dt \leqslant (2 + 8S^2)\int_0^S\|\partial_t u\|^2_{L^2(\omega(t))}\,dt. \tag{22}$$

Arguing as above, we have $\frac{d}{dt}E_0(u)(t) = -\|\partial_t u(t,x)\|^2_{L^2(\omega(t))}$. Using this property, inequalities (19) and (22), and the fact that $\phi_{|t=0} = u_{|t=0}$ and $\partial_t\phi_{|t=0} = \partial_t u_{|t=0}$, we deduce that $C_0E_0(u)(0) = C_0E_0(\phi)(0) \leqslant (2+8S^2)(E_0(u)(0) - E_0(u)(S))$, or rather $E_0(u)(S) \leqslant \big(1 - C_0/(2+8S^2)\big)E_0(u)(0)$. For S chosen sufficiently large, that is, for $\ell \in \mathbb{N}^*$ chosen sufficiently large, we thus have $E_0(u)(S) \leqslant \alpha E_0(u)(0)$ with $0 < \alpha < 1$.

Since ω is T-periodic and thus S-periodic, the above reasoning can be done on any interval $(kS,(k+1)S)$, yielding $E_0(u)((k+1)S) \leqslant \alpha E_0(u)(kS)$, for every $k \in \mathbb{N}$. Hence, we obtain $E_0(u)(kS) \leqslant \alpha^k E_0(u)(0)$.

For every $t \in [kS,(k+1)S)$, noting that $k = [t/S] > t/S - 1$, and that $\log(\alpha) < 0$, it follows that $\alpha^k < \frac{1}{\alpha}\exp(\frac{\ln(\alpha)}{S}t)$ and hence $E_0(u)(t) \leqslant E_0(u)(S) \leqslant \mu\exp(-\nu t)E_0(u)(0)$, for some positive constants μ and ν that are independent of u. □

3 Some examples and counter-examples

In the forthcoming examples, we shall consider several geometries in which the observation (or control) domain $\omega(t) = \{x \in \Omega \mid (t,x) \in Q\}$ is the rigid displacement in Ω of a fixed domain, with velocity v. Then the resulting observability property depends on the value of v with respect to the wave speed.

In all our examples, in the presence of a boundary we shall consider Dirichlet boundary conditions. In that case, generalized bicharacteristics behave as described in Section 1.3.1. We recall that, if parametrized by time t, the projections of the generalized bicharacteristics on the base manifold travel at speed one.

3.1 In dimension one

We consider $M = \mathbb{R}$ (Euclidean) and $\Omega = (0,1)$. The rays have a speed equal to 1. We set $I = (0,a)$, for some fixed $a \in (0,1)$, and we assume that the control domain $\omega(t)$ is equal to the translation of the interval I with fixed speed $v > 0$. We have, then, $\omega(t) = (vt, vt+a)$ as long as $t \in (0,(1-a)/v)$. When $\omega(t)$ touches the boundary, we assume that it is “reflected” after a time-delay $\delta \geqslant 0$, according to the following rule: if $(1-a)/v \leqslant t \leqslant (1-a)/v + \delta$ then $\omega(t) = (1-a,1)$. For larger times $t \geqslant (1-a)/v + \delta$ (and before the second reflection), the set $\omega(t)$ moves in the opposite direction with the same speed (see Figure 3).

This simple example is of interest as it exhibits that the control time depends on the value of the velocity v with respect to the wave speed (which is equal to 1 here). We denote by $T_0(v,a,\delta)$ the control time. With simple computations (see also Figure 3), we establish that

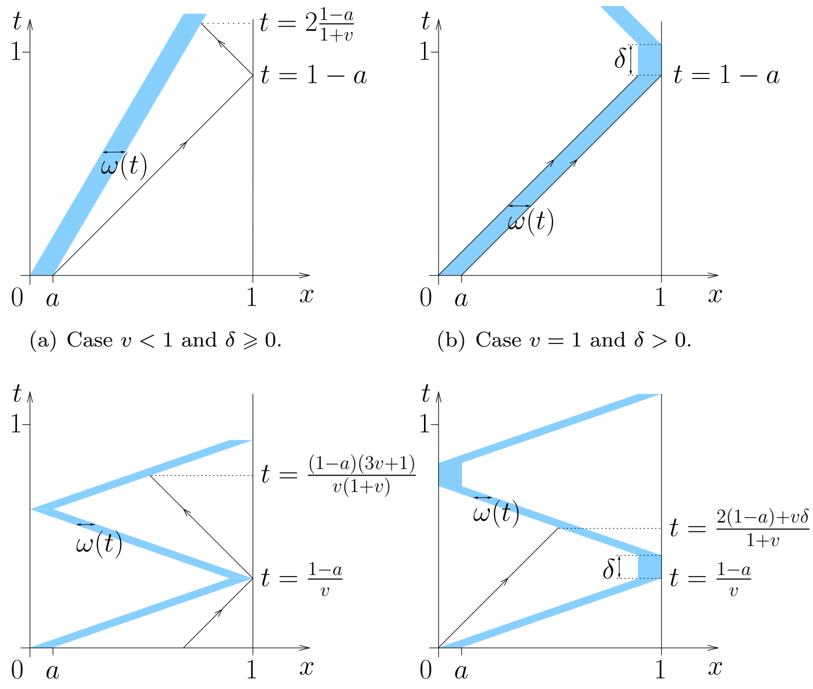
$$T_0(v,a,\delta) = \begin{cases} 2(1-a)/(1+v) & \text{if } 0 \leqslant v < 1 \text{ and } \delta \geqslant 0, \\ 1-a & \text{if } v = 1 \text{ and } \delta > 0, \\ (1-a)(3v+1)/\big(v(1+v)\big) & \text{if } v \geqslant 1 \text{ and } \delta = 0, \\ \big(2(1-a)+v\delta\big)(1+v) & \text{if } v > 1 \text{ and } \delta > 0. \end{cases}$$

Remark 3.1. Note that the control time $T_0(v,a,\delta)$ is discontinuous in v and δ. The control time is not continuous with respect to the domain Q as already mentioned in Remark 1.7.

3.2 A moving domain on a sphere

Let $M = \Omega = S^2$, the unit sphere of $\mathbb{R}^3$, be endowed with the metric induced by the Euclidean metric of $\mathbb{R}^3$. Let us consider spherical coordinates (θ,φ) on M, in which $\varphi = 0$ represents the horizontal plane (latitude zero), and θ is the angle describing the longitude along the equator. Let $a \in (0,2\pi)$ and $\varepsilon \in (0,\pi/2)$ be arbitrary. For $v > 0$, we set

$$\omega(t) = \{(\theta,\varphi) \mid |\varphi| < \varepsilon,\ vt < \theta < vt + a\},$$



(a) Case $v < 1$ and $\delta \geqslant 0$.

(b) Case $v = 1$ and $\delta > 0$.

(c) Case $v \geqslant 1$ and $\delta = 0$.

(d) Case $v > 1$ and $\delta > 0$.

Figure 3: Time-varying domains in dimension one.

for every $t \in \mathbb{R}$. The set $\omega(t)$ is a spherical square drawn on the unit sphere, with angular length equal to 2ε in latitude, and a in longitude, and moving along the equator with speed equal to v (see Figure 4). We denote by $T_0(v, a, \varepsilon)$ the control time as defined in Section 1.3.2.

For this example, an important fact is the following: every (geodesic) ray on the sphere propagates at speed one along a great circle, with half-period π. We thus have a simple description of all possible rays. Note that, as the radius is one, the speed coincides with the angular speed.

Proposition 3.2. *Let $a \in (0, 2\pi)$ and $\varepsilon \in (0, 1)$ be arbitrary. Then $T_0(v, a, \varepsilon) < +\infty$ except for a finite number of critical speeds $v > 0$. Moreover:*

- *$T_0(v, a, \varepsilon) \sim \frac{\pi - a}{v}$ as $v \to 0$;*
- *If $v > v_1 = (2\pi - a + 2\varepsilon)/(2\varepsilon)$ then $T_0(v, a, \varepsilon) < \infty$. If $v \to +\infty$ then $T_0(v, a, \varepsilon) \to \pi - 2\varepsilon$.*

Besides, if $v \in \mathbb{Q}$, then there exist $a_0 > 0$ and $\varepsilon_0 > 0$ such that $T_0(v, a, \varepsilon) = +\infty$ for every $a \in (0, a_0)$ and every $\varepsilon \in (0, \varepsilon_0)$.

Obtaining an analytic expression of T_0 as a function of (v, a, ε) seems to be very difficult.

Note that the asymptotics above still make sense if either $a > 0$ or $\varepsilon > 0$ are small. This shows that we can realize the observability property with a subset of arbitrary small Lebesgue measure (compare with Corollary 1.12).

Proof of Proposition 3.2. First, we observe the following. Consider a ray propagating along the equator (with angular speed 1), in the same direction as $\omega(t)$. If $v = 1$, depending on its initial

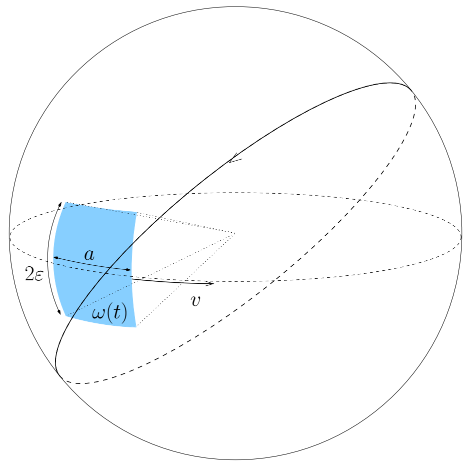


Figure 4: Time-varying domain on the unit sphere and a typical ray (great circle).

condition, this ray either never enters $\omega(t)$ or remains in it for all time. Hence, $T_0(v, a, \varepsilon) = +\infty$. In contrast, if $v \neq 1$, then, such a ray enters $\omega(t)$ for a time $0 \leqslant t < (2\pi)/|v-1|$, as $(2\pi)/|v-1| > (2\pi - a)/|v-1|$.

Second, we treat the cases v large and v small, and we compute the asymptotics of $T_0(v, a, \varepsilon)$ (in the argument, both a and ε are kept fixed).

Case $\boldsymbol{v}$ small. If $2\pi v < a$, then every ray goes full circle in a time shorter than that it takes for the domain to travel the distance a. It is then clear that every ray will have met $\omega(t)$ as soon as $\omega(t)$ has travelled halfway along the equator (up to the thickness of $\omega(t)$ and the travel time of the ray itself). In other words, we then have $\frac{\pi - a}{v} \leqslant T_0(v, a, \varepsilon) \leqslant \frac{\pi - a}{v} + 2\pi$ for $v < v_0 = a/(2\pi)$.

Case $\boldsymbol{v}$ large. If v grows to infinity, then the situation becomes intuitively as if we have a static control domain forming a strip of constant width $\varepsilon > 0$ around the equator. For such a strip, the control time is $\pi - 2\varepsilon$. More precisely, let us assume $(2\pi - a + 2\varepsilon)/v < 2\varepsilon$. Every ray entering the region $\{|\varphi| < \varepsilon\}$, spends a time at least equal to 2ε in this region. At worst, the control domain will have to travel the distance $2\pi - a + 2\varepsilon$ to "catch" this ray (going full circle and more than the longitudinal distance travelled by the ray itself). The condition $v > v_1 = (2\pi - a + 2\varepsilon)/(2\varepsilon)$ thus implies that all rays enter the moving open domain $\omega(t)$ within time $\pi - 2\varepsilon + 2(\pi + \varepsilon)/v$. Hence, $\pi - 2\varepsilon \leqslant T_0(v, a, \varepsilon) \leqslant \pi - 2\varepsilon + 2(\pi + \varepsilon)/v$.

Third, we consider the case $v_0 \leqslant v \leqslant v_1$. To get some intuition, we consider, in a first step, that a and ε are both very small, and thus consider $\omega(t)$ as point moving along the equator. According to the first observation we made above, let us consider a ray propagating along a great circle that is transversal to the equator. It meets the equator at times $t_k = t_0 + k\pi$, $k \in \mathbb{Z}$, for some t_0. If v is irrational then the set of positions of the "points" $\omega(t_k)$, given by $x(t_k) = \cos(vt_k)$ and $y(t) = \sin(vt_k)$ in the plane (x, y) containing the equator, is dense in the equator. Adding some thickness to $\omega(t)$, that is having $a > 0$ and $\varepsilon > 0$, we find that every ray encounters the moving open set $\omega(t)$ in a finite time if v is irrational. By a compactness argument we then obtain $T_0(v, a, \varepsilon) < \infty$ if v is irrational.

Fourth, considering again that $a > 0$ and $\varepsilon > 0$ are both very small, we shall now see that there do exist rays, transversal to the equator, that never meet the moving "point" $\omega(t)$ whenever $v \in \mathbb{Q}$.

Writing $v = p/q$ with p and q positive integers, the set of points reached by $(\cos(vt_k), \sin(vt_k))$ at times $t_k = t_0 + k\pi$, with $k \in \mathbb{Z}$, is finite. The following lemma yields a more precise statement.

Lemma 3.3. *Let p and q be two coprime integers. We have*

$$\left\{\frac{kp}{q}\pi \mod 2\pi \quad | \ k = 1, \ldots, 2q\right\} = \begin{cases} \left\{\frac{k}{q}\pi \quad | \ k = 1, \ldots, 2q\right\} & \textit{if } p \textit{ is odd,} \\ \left\{2\frac{k}{q}\pi \quad | \ k = 1, \ldots, q\right\} & \textit{if } p \textit{ is even (and } q \textit{ odd).} \end{cases}$$

Thus, if $v = p/q$, with p and q coprime integers, the points $(\cos(vt_k), \sin(vt_k))$ form the vertices of a regular polygon in the disk. There are exactly $2q$ (resp. q) such vertices if p is odd (resp. even). In this situation, it is always possible to find a ray transversal to the equator that never meets this set of vertices. Now, this phenomenon persists in the case $a > 0$ and $\varepsilon > 0$ if both are chosen sufficiently small. We have thus proven that, given $v \in \mathbb{Q} \cap [v_0, v_1]$, there exist $0 < a_0 < 2\pi$ and $0 < \varepsilon_0 < \pi/2$ such that $T_0(v, a, \varepsilon) = +\infty$ for all $a \in (0, a_0)$ and $\varepsilon \in (0, \varepsilon_0)$. Note also that if $a > 2\pi/q$ and $\varepsilon > 0$ then every ray meets $\omega(t)$ in some finite time. By a compactness argument we then obtain $T_0(v, a, \varepsilon) < \infty$. From that last observation, we infer that, given $a > 0$ and $\varepsilon > 0$ fixed, the set of rational velocities $v \in (v_0, v_1) \cap \mathbb{Q}$ for which $T_0(v, a, \varepsilon) = +\infty$ is finite. $\square$

Proof of Lemma 3.3. We note that $\{kp\pi/q \mod 2\pi \quad | \ k = 1, \ldots, 2q\} = \{kp\pi/q \mod 2\pi \quad | \ k \in \mathbb{Z}\}$. It thus suffices to prove the following two statements:

1. for p even: $\forall k' \in \{1, \ldots, q\}$, $\exists k \in \mathbb{Z}$, $\exists m \in \mathbb{Z}$ such that $2k' = kp + 2mq$.
2. for p odd: $\forall k' \in \{1, \ldots, 2q\}$, $\exists k \in \mathbb{Z}$, $\exists m \in \mathbb{Z}$ such that $k' = kp + 2mq$.

Since p and q are coprime, there exists $(a, b) \in \mathbb{Z}^2$ such that $ap + bq = 1$. Moreover, if (a, b) is a solution of that diophantine equation, then all other solutions are given by $(a + qn, b - pn)$, with $n \in \mathbb{Z}$. Multiplying by $2k'$, we infer that $2k' = 2k'ap + 2k'bq$, and the first statement above follows. For the second statement, we note that, if p is odd, then, changing b into $b - pn$ if necessary, we may assume that b is even, say $b = 2b'$. Then, multiplying by k', we infer that $k' = k'ap + 2k'b'q$, and the second statement follows. $\square$

Before moving on to the next example, we stress again that the peculiarity of the present example (unit sphere) is that all rays are periodic, with the same period 2π. The study of other Zoll manifolds would be of interest. The situation turns out to be drastically different in the case of a disk, due to "secular effects" implying a precession phenomenon, as we are now going to describe.

3.3 A moving domain near the boundary of the unit disk

Let $M = \mathbb{R}^2$ (Euclidean) and let $\Omega = \{(x, y) \in \mathbb{R}^2 \ | \ x^2 + y^2 < 1\}$ be the unit disk. Let $a \in (0, 2\pi)$ and $\varepsilon \in (0, 1)$ be arbitrary. We set, in polar coordinates,

$$\omega(t) = \{(r, \theta) \in [0, 1] \times \mathbb{R} \quad | \ 1 - \varepsilon < r < 1, \ vt < \theta < vt + a\},$$

for every $t \in \mathbb{R}$. The time-dependent set $\omega(t)$ moves at constant angular speed v, anticlockwise, along the boundary of the disk (see Figure 5). Its radial length is ε and its angular length is a.

Proposition 3.4. *The following properties hold:*

1. *Let $a \in (0, 2\pi)$ and $\varepsilon \in (0, 1)$ be arbitrary. We have $T_0(v, a, \varepsilon) < +\infty$, for every $v > v_0 = (2\pi + 2\varepsilon - a)/(2\varepsilon)$, and we have $T_0(v, a, \varepsilon) \sim 2 - 2\varepsilon$ as $v \to +\infty$.*

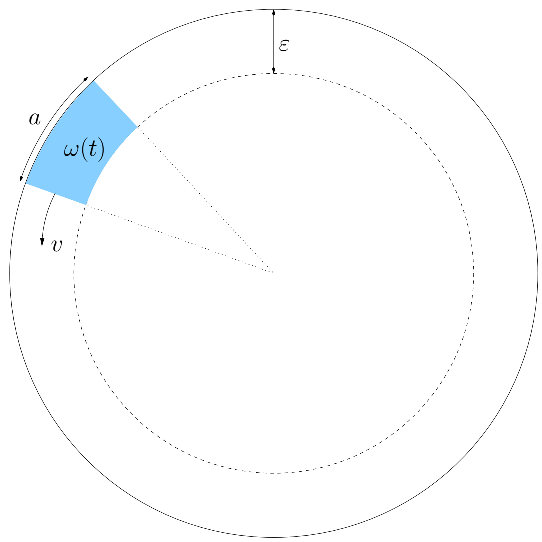


Figure 5: A time-varying domain on the unit disk

2. *If there exists* $n \in \mathbb{N} \setminus \{0,1\}$ *such that* $v \sin \frac{\pi}{n} \in \pi\mathbb{Q}$*, then there exist* $a_0 \in (0, 2\pi)$ *and* $\varepsilon_0 \in (0,1)$ *such that* $T_0(v, a, \varepsilon) = +\infty$ *for all* $a \in (0, a_0)$ *and* $\varepsilon \in (0, \varepsilon_0)$*.*

3. *For every* $v \geqslant 1$*, for every* $a \in (0, 2\pi)$*, there exists* $\varepsilon_0 > 0$ *such that* $T_0(v, a, \varepsilon) = +\infty$*, for every* $\varepsilon \in (0, \varepsilon_0)$*.*

4. *For every* $v \geqslant 0$ *and* $a \in (0, \pi)$*, there exists* $\varepsilon_0 \in (0,1)$ *such that* $T_0(v, a, \varepsilon) = +\infty$*, for every* $\varepsilon \in (0, \varepsilon_0)$*.*

As for the case of the sphere presented in Section 3.2, obtaining an analytic expression of T_0 as a function of (v, a, ε) seems a difficult task.

The fact that $T_0(v, a, \varepsilon) = +\infty$ provided that $a > 0$ and $\varepsilon > 0$ are chosen sufficiently small is in strong contrast with the case of the sphere. This is due to the fact that, in the disk, the structure of the rays is much more complex: there are large families of periodic and almost-periodic rays. The ray drawn in Figure 7 produces some sort of "secular effect", itself implying a precession whose speed can be tuned to coincide with the speed v of $\omega(t)$, provided $v \geqslant 1$. We shall use this property in the proof.

We stress that for the third property we do not need to assume that a is small. Actually, a is any element of $(0, 2\pi)$. If a is close to 2π, then $\omega(t)$ is almost a ring located at the boundary, moving with angular speed v, with a "hole". As this hole moves around with speed v there is a ray that periodically hits the boundary and reflects from it exactly at the hole position.

Proof of Proposition 3.4. For $a \in (0, 2\pi)$ and $\varepsilon \in (0,1)$ fixed, if v is very large, then the situation gets close to that of a static control domain which is a ring of width ε, located at the boundary of the disk. For this static domain the control time is $2 - 2\varepsilon$. In fact, all rays enter the region $\Omega_\varepsilon = \{1 - \varepsilon < r < 1\}$ and the shortest time spent there is 2ε. During such time the angular distance travelled by the ray is less than 2ε. Hence, if $(2\pi + 2\varepsilon - a)/v < 2\varepsilon$, one knows for sure that the ray will be "caught" by the moving open set $\omega(t)$ before it leaves Ω_ε. Thus, for $v > v_0 = (2\pi + 2\varepsilon - a)/(2\varepsilon)$ all rays enter $\omega(t)$ in finite time. Moreover, we have $2 - 2\varepsilon \leqslant T_0(v, a, \varepsilon) \leqslant 2 - 2\varepsilon + (2\pi + 2\varepsilon)/v$. This yields the announced asymptotics for $T_0(v, a, \varepsilon)$.

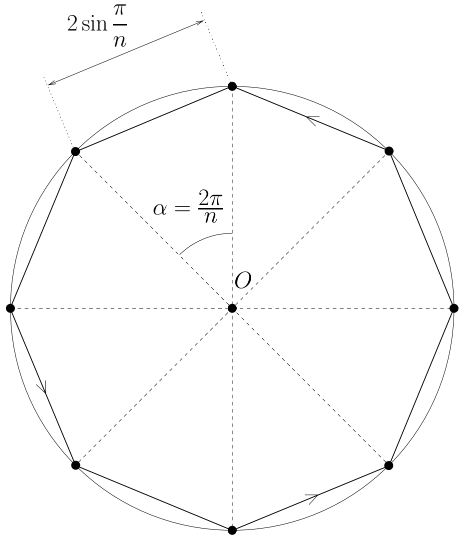
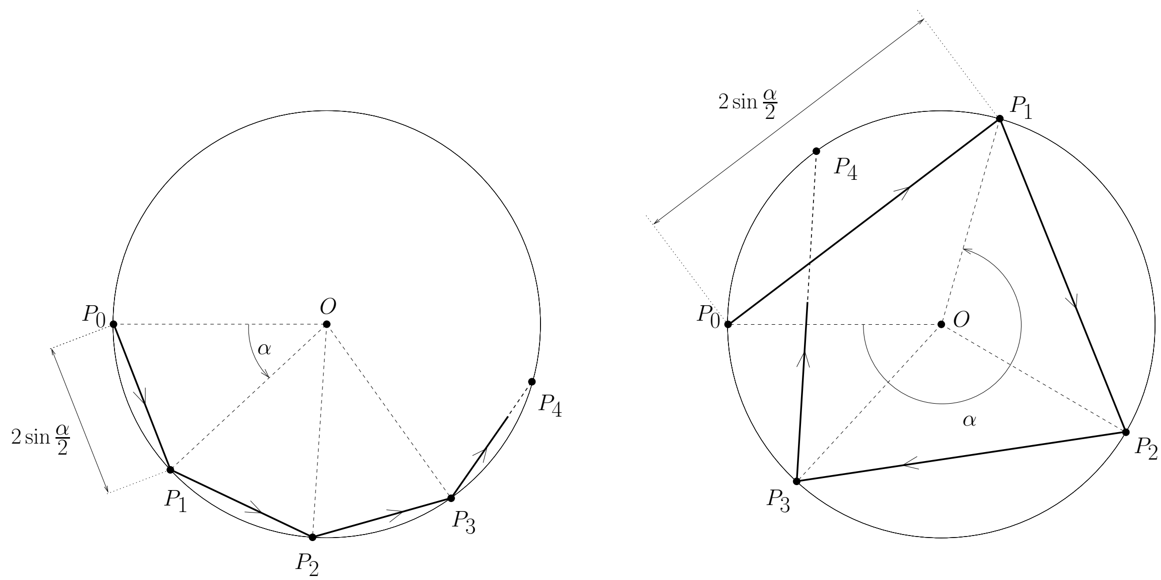


Figure 6: A periodic ray yielding a regular polygon

Let us now investigate the three cases where $T_0(v, a, \varepsilon) = +\infty$ as stated in the proposition. For the sake of intuition, it is simpler to first assume that $\varepsilon > 0$ and $a > 0$ are very small, and hence, that $\omega(t)$ is close to being a point moving along the boundary of the disk, given by $(\cos(vt), \sin(vt))$.

Let us then consider, as illustrated in Figure 6, *periodic* rays propagating "anticlockwise" in the disk (with speed equal to 1), and reflecting at the boundary of the disk according to Section 1.3.1, that is, according to geometrical optics. The trajectory of such rays forms a regular polygon with vertices at the boundary of the unit disk. Let $n \geqslant 2$ be the number of vertices. For $n = 2$, the ray travels along a diameter of the disk, and passes through the origin; it is 4-periodic. For $n = 3$, the trajectory of the ray forms an equilateral triangle centered at the origin; etc. The length of an edge of such a regular polygon with $n \geqslant 2$ vertices is equal to $2\sin\frac{\pi}{n}$. This means that there exists $t_0 \in \mathbb{R}$ such that this ray reaches the boundary at times $t_k = t_0 + 2k\sin\frac{\pi}{n}$. Hence, if $2v\sin\frac{\pi}{n} = \frac{2p\pi}{q}$ with p and q positive integers, then the moving point $(\cos(vt), \sin(vt))$, taken at times t_k ranges over a finite number of points of $\partial\Omega$. Therefore, there exists a periodic ray with n vertices never meeting $\omega(t)$. This property remains clearly true for values of $a > 0$ and of $\varepsilon > 0$ chosen sufficiently small. This show the second statement of the proposition.

Let us now consider a ray propagating in the disk, as drawn in Figures 7(a) and 7(b), and reflecting at the boundary at consecutive points P_k, $k \in \mathbb{N}$. Denote by O the center of the disk, and by α the oriented angle $\widehat{P_0OP_1}$. If $0 < \alpha < \pi$, then the ray appears to be going "anticlockwise" as in Figure 7(a); if $\alpha = \pi$ then the ray bounces back and forth on a diameter of the disk; if $\pi < \alpha < 2\pi$, then the ray appears to be going "clockwise" as in Figure 7(b). In any case, the distance P_0P_1, and more generally P_kP_{k+1}, is equal to $2\sin(\alpha/2)$. Since the speed of the ray is equal to 1, the ray starting from P_0 at time $t = 0$ reaches the point P_1 at time $2\sin(\alpha/2)$, the point P_k at time $t_k = 2k\sin(\alpha/2)$, etc. Let $t \mapsto P(t)$ be the curve propagating *anticlockwise* along the unit circle, with constant angular speed, passing exactly through the points P_k at time t_k. Its



(a) "anticlockwise", $0 < \alpha \leqslant \pi$.

(b) "Clockwise", $\pi \leqslant \alpha < 2\pi$.

Figure 7: Rays propagating "anticlockwise" and "clockwise".

angular speed is then

$$w_P(\alpha) = \frac{\alpha}{2\sin(\alpha/2)},$$

and we call it the *precession speed*. This is the speed at which the discrete points P_k propagate "anticlockwise" along the unit circle. Now, if the set $\omega(t)$ has the angular speed $v = w_P(\alpha)$ (for some $\alpha \in (0, 2\pi)$), then there exists rays, as in Figures 7(a) or 7(b), that never meet $\omega(t)$, if $a \in (0, 2\pi)$, provided that $\varepsilon > 0$ is chosen sufficiently small. Since the function $\alpha \mapsto w_P(\alpha)$ is monotone increasing from $(0, 2\pi)$ to $(1, \infty)$, it follows that, for every $v \in (1, \infty)$, there exists $\alpha \in (0, 2\pi)$ such that $v = w_P(\alpha)$, and therefore $T_0(v, a, \varepsilon) = +\infty$ provided that ε is chosen sufficiently small. For $v = 1$, there exists a gliding ray that never meets $\omega(t)$. This can be seen as the limiting case $\alpha \to 0$, as rays can be concentrated near the gliding ray. We thus have proven the third statement of the proposition.

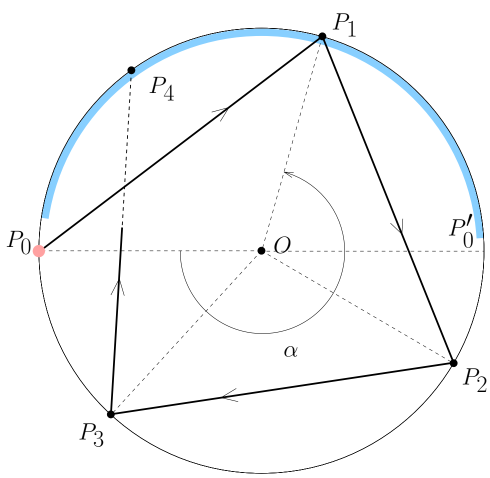
Now, still working with the configurations drawn in Figure 7(b), for $\alpha \in (\pi, 2\pi)$, let $P(t)$ be the curve propagating *anticlockwise* along the unit circle, with constant angular speed, passing successively through P_0 at time 0, through P_2 at time $4\sin(\alpha/2)$, and P_{2k} at time $2k\sin(\alpha/2)$. Its angular speed is given by

$$w_P(\alpha) = \frac{2\alpha - 2\pi}{4\sin(\alpha/2)} = \frac{\alpha - \pi}{2\sin(\alpha/2)}.$$

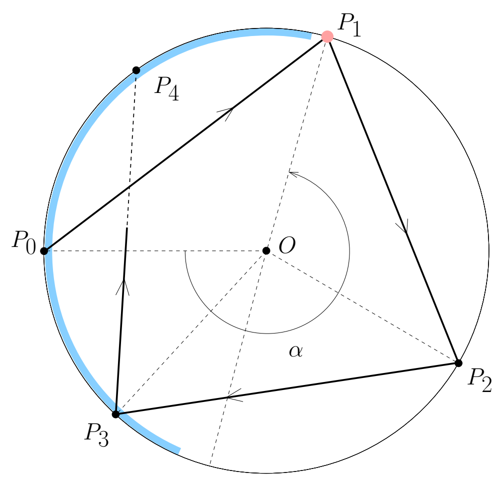
The function $\alpha \mapsto w_P(\alpha)$ is monotone increasing from $(\pi, 2\pi)$ to $(0, +\infty)$. If the set $\omega(t)$, with $a \in (0, \pi)$ and $\varepsilon > 0$ small, is initially (at time 0) located between the points P_0 and P_0' its diametrically opposite point, and if $v = w_P(\alpha)$, then the ray drawn in Figure 7(b) never meets $\omega(t)$. This is illustrated in Figure 8. We have thus proven the last statement of the proposition. $\square$

Remark 3.5.

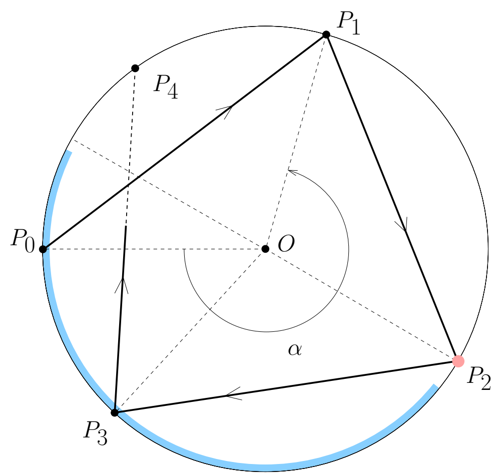
1. It is interesting to note that, even for domains such that a is close to 2π, the $t - GCC$ property fails if $v > 1$ and if ε is chosen too small. This example is striking, because in that case, if the control domain were static, then it would satisfy the usual GCC (this is true as soon as $a > \pi$). This example shows that, when considering a control domain satisfying the GCC, then, when making it move, the t-GCC property may fail. However, this example is a domain moving faster than the actual wave speed. This is rather non physical.



(a) $t = 0$.



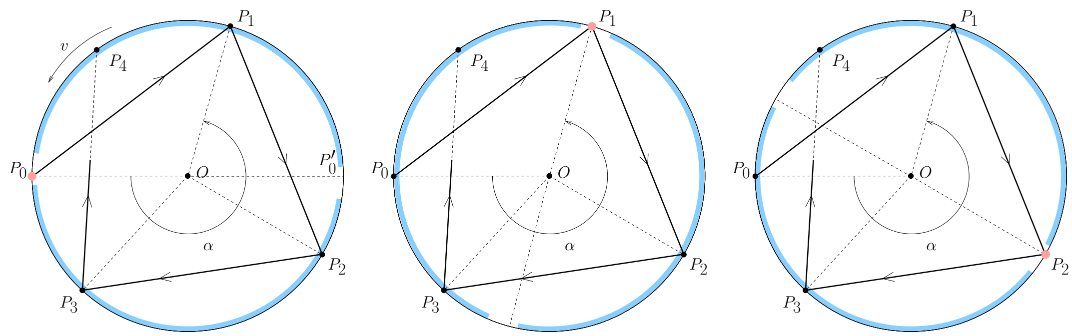
(b) $t = 2\sin(\alpha/2)$.



(c) $t = 4\sin(\alpha/2)$.

Figure 8: Illustration of property 4 of Proposition 3.4.

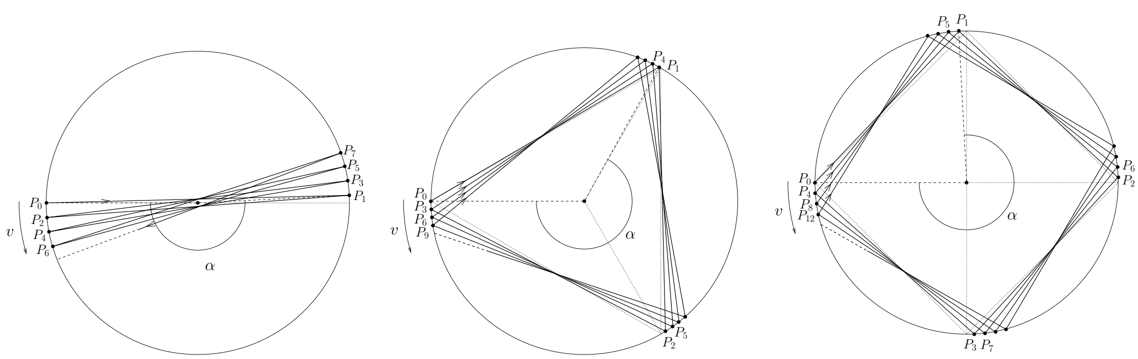
2. For the fourth property of the previous proposition we obtain a moving open set $\omega(t)$ with an "angular measure" that is less than π, that is $0 < a < \pi$ (see Figure 8). In fact, if one allows for $\omega(t)$ to be not connected, but rather the union of two connected components, for any velocity $v \in (0, +\infty)$ we can exhibit moving sets whose "angular measure" is a close as



(a) $t = 0$. (b) $t = 2\sin(\alpha/2)$. (c) $t = 4\sin(\alpha/2)$.

Figure 9: Case of a non connected moving open set $\omega(t)$ moving anticlockwise, non statisfying the t-GCC with yet a very large "angular" measure.

one wants to 2π and yet the t-GCC does not hold. This is illustrated in Figure 9.



(a) $n = 2$, $\pi < \alpha < \pi + \delta$. (b) $n = 3$, $4\pi/3 < \alpha < 4\pi/3 + \delta$. (c) $n = 4$, $3\pi/2 < \alpha < 3\pi/2 + \delta$.

Figure 10: Cases of slow precession speeds with $0 < \alpha - 2\pi\frac{n-1}{n} < \delta$, with δ small and $n \geqslant 2$, that is, with a trajectory "close" to that of a periodic ray that forms a regular polygon (see Figure 6).

3. If v cannot be chosen as large as desired (for physical reasons), Proposition 3.4 states that the t-GCC does not hold true if $a > 0$ and $\varepsilon > 0$ are too small. As shown in the proof above, this lack of observability is due to a secular effect caused by geodesics whose trace at the boundary produces a pattern that itself varies in time, with a precession speed that can be tuned to match that of the control domain. In fact, a precession speed can be obtained as slow as one wants if α is chosen such that $\pi < \alpha < \pi + \delta$ with $\delta > 0$ small. This is illustrated in Figure 10(a). If one is close to regular polygons, as illustrated in Figure 10, one obtains a precession pattern, that can be used to deduce other families of examples of moving domains $\omega(t)$ with multiple connected components with velocities $v > 0$ that do not satisfy the t-GCC.

4. Proposition 3.4 has been established for the domain drawn in Figure 5, sliding anticlockwise

along the boundary with a constant angular speed. Other situations can be of interest: we could allow the domain to move with a nonconstant angular speed. For instance, we could allow the domain to move anticlockwise within a certain horizon of time, and then clockwise. This would certainly improve the observability property. A situation that can be much more interesting in view of practical issues is to let the angular speed of the control domain evolve according to $v(t) = v + \beta \sin(\gamma t)$ (with $\beta > 0$ small), that is, with a speed oscillating around a constant value v. We expect that such a configuration, with an appropriate choice of coefficients, will yield the observability property to be more robust, by avoiding the situation described in the second property of Proposition 3.4 (non-observability for a dense set of speeds).

We now state a positive result. For $a \in (0, 2\pi)$ and $\varepsilon \in (0, 1)$, assume now that the domain $\omega(t)$ is given by $\omega(t) = \{(r, \theta) \in [0, 1] \times \mathbb{R} \; ; \; 1 - \varepsilon < r < 1, \; \theta_0(t) < \theta < a + \theta_0(t)\}$, with

$$\theta_0(t) = \begin{cases} 0 & \text{if } 0 \leqslant t < t_0, \\ v(t - t_0) & \text{if } t_0 \leqslant t, \end{cases}$$

for some $t_0 > 0$ and $v > 0$. We set $Q = \cup_{t \geqslant 0} \omega(t)$. In this configuration, at first, the domain is still, and then one lets it move.

Proposition 3.6. *If $4\pi/5 < a < \pi$, $t_0 > 2\pi$, then there exists $0 < v < 1$ such that $T_0(Q, \Omega) < \infty$.*

Remark 3.7.

1. The important aspect of this result lies in the following facts. First, if at rest, the observability set does not satisfy the geometric control condition; hence, its motion is crucial for the t-GCC to hold. Second, the motion is performed at a velocity v that is less than that of the wave speed; we thus have a physically meaningful example.

2. The result is not optimal as we do not exploit the thickness ε of the domain $\omega(t)$ in the proof.

3. It would be interesting to further study this "stop-and-go" strategy and see how small the value of $a > 0$ can be chosen.

Proof of Proposition 3.6. Let $0 < t < t_0$; then $\omega(t) = \omega(0)$ is still. First, we consider the ray associated with $0 \leqslant \alpha < a$, or symmetrically $2\pi - a < \alpha \leqslant 2\pi$, then the movement of the successive points P_k, $k \in \mathbb{N}$, is anticlockwise, or clockwise, respectively; see Figure 7. Depending on the case considered we denote $\beta = \alpha$ or $\beta = 2\pi - \alpha$. The above condition thus reads better as follows $0 \leqslant \beta < a$. In both cases, the (unsigned) angular distance between two points is precisely β. The successive points P_k, $k \in \mathbb{N}$, thus end up meeting $\omega(0)$ in finite time. (The case $\beta = 0$ coincides with a gliding ray that has angular speed 1; it thus meet $\omega(0)$ in finite time.) Let us consider $\beta \neq 0$. The maximal number of steps it takes for any ray associated with β to enter $\omega(0)$ is then $\lfloor \frac{2\pi - a}{\beta} \rfloor + 1$ yielding a maximal time $T_0(\alpha) = 2\sin(\beta/2)\big(\lfloor \frac{2\pi-a}{\beta} \rfloor + 1\big)$, as the time lapse between two points P_k is $2\sin(\beta/2)$. Here, the notation $\lfloor . \rfloor$ stands for the usual floor function. We thus need $T_0 > \max\big(2\pi - a, \sup_{0<\beta<a} T_0(\alpha)\big)$. The value $2\pi - a$ accounts for the gliding rays ($\beta = 0$). Here, we give a crude upper bound for $T_0(\alpha)$ observing that

$$T_0(\alpha) \leqslant \frac{\sin(\beta/2)}{\beta/2}(2\pi - a + \beta) \leqslant (2\pi - a + \beta) < 2\pi.$$

We thus see that if we choose $t_0 > 2\pi$ then all the rays associated with the angle $0 \leqslant \beta < a$ enter $\omega(0)$ for $0 \leqslant t < t_0$.

Second, we consider $t \geqslant t_0$ and we are left only with the rays that are associated with $a \leqslant \alpha \leqslant 2\pi - a$. For these rays we consider the two sequences of points $(P_{2k})_k$ and $(P_{2k+1})_k$. The time lapse for a ray to go from one point to the consecutive point in these two sequences is $4\sin(\alpha/2)$ and this is associated with the (signed) angle $2(\alpha - \pi)$. In fact, as $a > \pi/2$, if $a \leqslant \alpha \leqslant \pi$, then both sequences move clockwise and if $\pi \leqslant \alpha \leqslant 2\pi - a$, then both sequences move anticlockwise.

If $v > 0$ is the angular speed of $\omega(t)$ for $t \geqslant t_0$ then we require $2(\alpha - \pi) < 4\sin(\alpha/2)v < a + 2(\alpha - \pi)$, that is

$$\frac{(\alpha - \pi)}{2\sin(\alpha/2)} < v < \frac{a + 2(\alpha - \pi)}{4\sin(\alpha/2)}, \qquad \text{for } a \leqslant \alpha \leqslant 2\pi - a. \tag{23}$$

Observe that $a + 2(\alpha - \pi) > 0$ as $a > \pi/2$. The left inequality in (23) is necessary as it implies that the anticlockwise moving open set $\omega(t)$ will be faster than the two sequences given above, a necessary condition to be able to catch points in thoses sequences. In particular, this necessary condition is clearly filled if the sequences move clockwise, that is if $a \leqslant \alpha \leqslant \pi$. The right inequality in (23) expresses that $\omega(t)$ will not turn too fast and then miss the discrete sequences of points. In fact, during a time interval of length $4\sin(\alpha/2)$ the relative angular displacement of the sequence and the moving set $\omega(t)$ is $\ell = 4\sin(\alpha/2)v - 2(\alpha - \pi)$ and with (23) we have $0 < \ell < a$. This expresses that the sequence points cannot be missed.

As both bounds of (23) are increasing functions for $\alpha \in (a, 2\pi - a)$ we obtain the following sufficient condition

$$\frac{(\pi - a)}{2\sin(a/2)} < v < \frac{3a - 2\pi}{4\sin(a/2)}.$$

We see that it can be satisfied if $a > 4\pi/5$. Observing that $a \mapsto h(a) = \frac{(3a-2\pi)}{4\sin(a/2)}$ increases on $(0, \pi]$ and $0 < h(\pi) = \pi/4 < 1$ we see that the found admissible velocities are such that $0 < v < 1$. $\square$

3.4 A moving domain in a square

Let $M = \mathbb{R}^2$ (Euclidean) and $\Omega = (0,1)^2$, be the unit square. We recall that, as discussed in Remark 1.9–(2), the statement of Theorem 1.8 is still valid in the square, because the generalized bicharacteristic flow is well defined.

Let $a \in (0, 1)$. We consider the fixed domain $\tilde{\omega}_0 = (-a, a)^2$, a square centered at the origin $(0, 0)$, and we set $\omega_0 = \tilde{\omega}_0 \cap \Omega$ (see Figure 11). Since there are periodic rays, bouncing back and forth between opposite sides of the square, that remain away from ω_0, the GCC does not holds true for ω_0, and the wave equation cannot be observed from the domain ω_0 in the sense of (5).

Now, for a given $T > 0$, consider a continuous path $t \in [0, T] \mapsto (x(t), y(t))$ in the closed square $[0, 1]^2$, with $(x(0), y(0)) = (0, 0)$. We set

$$\tilde{\omega}(t) = (x(t) - a, x(t) + a) \times (y(t) - a, y(t) + a) \quad \text{and} \quad \omega(t) = \tilde{\omega}(t) \cap \Omega.$$

To avoid the occurrence of periodic rays, as described above, that never meet $\omega(t)$ a necessary condition for the t-GCC to hold true is

$$[a, 1 - a] \subset x([0, T]) \quad \text{and} \quad [a, 1 - a] \subset y([0, T]).$$

Let us assume that the point $(x(t), y(t))$ moves precisely along the boundary of the square $[0, 1]^2$, anticlockwise, and with a constant speed v. The path $t \mapsto (x(t), y(t))$ only exhibits singularities when reaching a corner of the square $[0, 1]^2$, where the direction of the speed is discontinuous (see Figure 11).

We denote by $T_0(v, a)$ the control time.

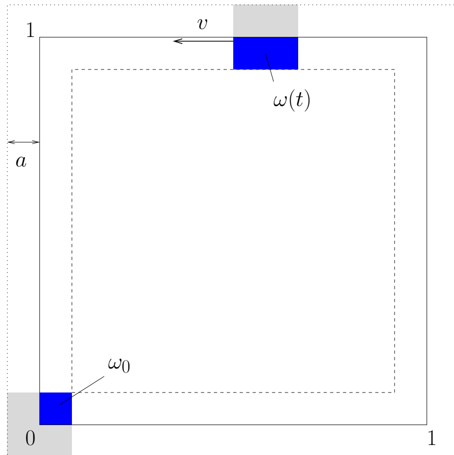


Figure 11: Time-varying domain in the square $(0,1)^2$.

Proposition 3.8. *We have the following two results:*

1. *Let $a \in (0,1/2)$ be arbitrary. For $v > v_0 = (2-a)/a$ we have $T_0(v,a) < +\infty$, and moreover $T_0(v,a) \sim \max(\sqrt{2}(1-2a),0)$ as $v \to +\infty$.*

2. *If $v \in \cup_{(p,q)\in\mathbb{N}} \sqrt{p^2+q^2}\mathbb{Q}$, there exists $a_0 > 0$ such that $T_0(v,a) = +\infty$ if $a \in (0,a_0)$.*

Proof of Proposition 3.8. The argument for the first property is the same as that developed in the other examples. For $v > 0$ large, the situation becomes intuitively as if we have a static control domain that forms a a-thick strip along the boundary of the square. For this case, the geometric control time is $\sqrt{2}(1-2a)$ if $a \in (0,1/2)$ and 0 otherwise. More precisely, if a ray enters this strip, it remains in it at least for a time $2a$. During such time, it travels at most a lateral distance equal to $2a$ (wave speed is one). If during that time the control domain goes all around and travels also the additional $2a$ distance, we can be sure that this ray will be "caught" by the moving domain. For $v > v_0 = (4+2a)/2a = (2-a)/a$, the control time can thus be estimated by $\max(\sqrt{2}(1-2a),0) \leqslant T_0(v,a) \leqslant \max(\sqrt{2}(1-2a),0) + (4+2a)/v$. Hence, the announced asymptotics.

For the second property, as for the other examples, considering a small at first, and thus the set $\omega(t)$ to be a simple point running along the boundary, greatly helps intuition. We start by considering 2-periodic rays that bounce back and forth between two opposite sides of the square. They reflect at boundaries at times $t_k = t_0 + k$, $k \in \mathbb{Z}$, for some $t_0 \in \mathbb{R}$. If $v = \frac{p}{q}$ is rational, then the positions of the "moving point" $\omega(t)$ at times t_k range over a finite number of points. One can thus easily identify 2-periodic rays that never meet that moving point. This property remains true if $a > 0$ is chosen sufficiently small.

Let us consider more general periodic rays. All rays propagating in the square can be described as follows. Let $(x_0,y_0) \in [0,1]^2$ be arbitrary. Let us consider a ray $t \mapsto (x(t),y(t))$ starting from (x_0,y_0) at time $t = t_0$, with a slope $\tan(\alpha) \in \overline{\mathbb{R}}$, for some $\alpha \in (-\pi,\pi]$. Setting $c = \cos(\alpha)$ and $s = \sin(\alpha)$, we define $\tilde{x}(t) = x_0 + (t-t_0)c$ and $\tilde{y}(t) = y_0 + (t-t_0)s$. Then, for times $t \in \mathbb{R}$ such that $|t-t_0|$ is small (possibly only if $t \leqslant t_0$ or $t \geqslant t_0$), the ray is given by $t \mapsto (\tilde{x}(t),\tilde{y}(t))$. Introducing $\hat{x}(t),\hat{y}(t) \in [0,2)$ such that $\hat{x}(t) = \tilde{x}(t) \mod 2$ and $\hat{y}(t) = \tilde{y}(t) \mod 2$ it can be seen, by "developing the square" by means of plane symmetries, that the ray is given by

$$x(t) = \begin{cases} \hat{x}(t) & \text{if } 0 \leqslant \hat{x}(t) \leqslant 1, \\ 2 - \hat{x}(t) & \text{if } 1 < \hat{x}(t) < 2, \end{cases} \qquad y(t) = \begin{cases} \hat{y}(t) & \text{if } 0 \leqslant \hat{y}(t) \leqslant 1, \\ 2 - \hat{y}(t) & \text{if } 1 < \hat{y}(t) < 2. \end{cases} \tag{24}$$

A ray is periodic if and only if $\tan(\alpha) = p/q \in \mathbb{Q} \cup \{+\infty, -\infty\}$, with p and q relatively prime integers (including the case $q = 0$). The period is equal to $2\sqrt{p^2 + q^2}$. In this case, we have $c = q/\sqrt{p^2 + q^2}$ and $s = p/\sqrt{p^2 + q^2}$. Such a ray reflects from the boundary of the square at times t in the union of the following (possibly empty) subsets of $\mathbb{R}$:

$$A_x = t_0 + \{(k - x_0)/c \mid k \in \mathbb{Z}\}, \qquad A_y = t_0 + \{(k - y_0)/s \mid k \in \mathbb{Z}\}.$$

The set $M = M(x_0, y_0, p, q)$ of associated points where this ray meets the boundary is then finite and independent of t_0.

If now $v = r\sqrt{p^2 + q^2}$ with $r \in \mathbb{Q}^+$, at times $t \in A_x \cup A_y$, the "moving point" $\omega(t)$ ranges over a finite set $L = L(x_0, y_0, t_0, p, q, r)$ of points on the boundary of the square, as the accumulated distance travelled along the boundary of the square $(0,1)^2$ is of the form $d_k = t_0 v + (k - x_0) r(p^2 + q^2)/q$ or $d'_k = t_0 v + (k - y_0) r(p^2 + q^2)/p$ and simply needs to be considered modulo 4. Adjusting the value of the time t_0, we can enforce $M \cap L = \emptyset$. Hence, the associated ray never meets the moving point $\omega(t)$. Finally, as the number of points is finite, this property remains true if $a > 0$ is chosen sufficiently small. $\square$

Remark 3.9. If $\tan(\alpha) \in \mathbb{R} \setminus \mathbb{Q}$, then the set of points at which the corresponding ray reflects at the boundary $\partial\Omega$ is dense in $\partial\Omega$. In fact at such a point, using the parametrization given in the proof above, we have either $\tilde{x}(t) = x_0 + (t - t_0)\cos(\alpha) \in \mathbb{Z}$, or $\tilde{y}(t) = y_0 + (t - t_0)\sin(\alpha) \in \mathbb{Z}$. For instance, if $\tilde{x}(t) = 2k \in \mathbb{Z}$, that is, $x(t) = 0$, meaning that we consider point on the left-hand-side vertical side of the square, then the corresponding $\tilde{y}(t)$ satisfies $\tilde{y}(t) = (y_0 - x_0\tan(\alpha)) + 2k\tan(\alpha)$. Using the fact that the set that $\{2k\beta \mod 2 \mid k \in \mathbb{Z}\}$ is dense in $[0, 2]$ if and only if $\beta \in \mathbb{R}\setminus\mathbb{Q}$, we conclude that $\hat{y}(t) = \tilde{y}(t) \mod 2$ is dense in $[0, 2]$ and thus $y(t)$ is dense in $[0, 1]$ considering (24). From this density, we deduce that the analysis in the case $\tan(\alpha) \in \mathbb{R} \setminus \mathbb{Q}$ may be quite intricate.

3.5 An open question

Let Ω be a domain of $\mathbb{R}^d$. Let ω_0 be a small disk in Ω, of center $x_0 \in \Omega$ and of radius $\varepsilon > 0$. Let $v > 0$ arbitrary. Given a path $t \mapsto x(t)$ in Ω, we define $\omega(t) = B(x(t), \varepsilon)$ (an open ball), with $x(0) = x_0$. We say that the path $x(\cdot)$ is admissible if $\omega(t) \subset \Omega$ for every time t. We raise the following question:

> Do there exist $T > 0$ and an admissible $\mathscr{C}^1$ path $t \mapsto x(t)$ in Ω, with speed less than or equal to v, such that (Q, T) satisfies the t-GCC? $(\star)$

Here, we have set $Q = \{(t, x) \in \mathbb{R} \times \Omega \mid t \in \mathbb{R},\ x \in \omega(t)\}$. Of course, many variants are possible: the observation set is not necessarily a ball, its velocity may be constant or not. The examples of Sections 3.1–3.4 have shown that addressing the question $(\star)$ is far from obvious. Assumptions on the domain Ω could be made; for instance, one may wonder whether ergodicity of Ω may help or not.

Note that, above in $(\star)$, we restrict the speed of $t \mapsto x(t)$. In fact, using arguments as in the beginning of both the proofs of Proposition 3.4 and Proposition 3.8, if $t \mapsto x(t)$ is periodic and if $\cup_{t\in\mathbb{R}} B(x(t), \varepsilon)$ satisfies the "static" Geometric Control Condition, then for a sufficiently large speed of the moving point, the set Q satisfies the t-GCC. An estimate of the minimal speed can be derived from the inner diameter of $\cup_{t\in\mathbb{R}} B(x(t), \varepsilon)$.

4 Boundary observability and control

In this section, we briefly extend our main results to the case of boundary observability. We consider the framework of Section 1.2, and we assume that $\partial\Omega$ is not empty. We still consider the wave equation (3), and we restrict ourselves, for simplicity, to Dirichlet conditions along the boundary.

Let R be an open subset of $\mathbb{R}\times\partial\Omega$. We set

$$\Gamma(t) = \{x \in \partial\Omega \ \mid \ (t,x) \in R\}.$$

We say that the Dirichlet wave equation is observable on R in time T if there exists $C > 0$ such that

$$C\|(u(0,\cdot), u_t(0,\cdot))\|^2_{H^1_0(\Omega)\times L^2(\Omega)} \leqslant \int_0^T \int_{\Gamma(t)} \left|\frac{\partial u}{\partial n}(t,x)\right|^2 \, d\mathcal{H}^{n-1}\, dt, \tag{25}$$

for all solutions of (3) with Dirichlet boundary conditions. Here, $\mathcal{H}^{n-1}$ is the $(n-1)$-Hausdorff measure.

In the static case (that is, if $\Gamma(t) \equiv \Gamma$ does not depend on t), the observability property holds true as soon as (Γ, T) satisfies the following GCC (see [2, 4]):

> *Let $T > 0$. The open set $\Gamma \subset \partial\Omega$ satisfies the Geometric Control Condition if the projection onto M of every (compressed) generalized bicharacteristic meets Γ at a time $t \in (0,T)$ at a nondiffractive point.*

Recall the definition of nondiffractive points given in Section 1.3.1. The t-GCC is then defined similarly to Definition 1.6.

Definition 4.1. Let R be an open subset of $\mathbb{R}\times\partial\Omega$, and let $T > 0$. We say that (R,T) satisfies the *time-dependent Geometric Control Condition* (in short, t-GCC) if every compressed generalized bicharacteristic ${}^b\gamma : \mathbb{R} \ni s \mapsto (t(s), x(s), \tau(s), \xi(s)) \in {}^bT^*Y \setminus E$ is such that there exists $s \in \mathbb{R}$ such that $t(s) \in (0,T)$ and $(t(s), x(s)) \in R$ and ${}^b\gamma(s) \in H \cup G \setminus G_d$ (a hyperbolic point or a glancing yet nondiffractive point).

We say that R satisfies the t-GCC if there exists $T > 0$ such that (R,T) satisfies the t-GCC.

The control time $T_0(R,\Omega)$ is defined by

$$T_0(R,\Omega) = \inf\{T > 0 \ \mid \ (R,T) \text{ satisfies the } t\text{-GCC}\},$$

with the agreement that $T_0(R,\Omega) = +\infty$ if R does not satisfy the t-GCC.

Theorem 4.2. *Let R be an open subset of $\mathbb{R}\times\partial\Omega$ that satisfies the t-GCC. Let $T > T_0(R,\Omega)$. We assume moreover that no (compressed) generalized bicharacteristic has a contact of infinite order with $(0,T)\times\partial\Omega$, that is, $G^\infty = \emptyset$. Then, the observability inequality (25) holds.*

Proof. The proof is similar to the proof of Theorem 1.8 done in Section 2.1. We just point out that the set of invisible solutions is defined by

$$N_T = \big\{u \in H^1((0,T)\times\Omega) \ \mid \ u \text{ is a Dirichlet solution of (3), and } \chi_R \frac{\partial u}{\partial n} = 0\big\}.$$

If $u \in N_T$ and $\rho \in T^*((0,T)\times\Omega)$, we wish to prove that u is smooth at ρ. Let ${}^b\gamma(s) = (x(s), t(s), \xi(s), \tau(s))$ be the compressed bicharacteristic that originates from ρ (at $s = 0$). There exists $s_0 \in \mathbb{R}$ such that $t(s_0) \in (0,T)$ and $(t_0, x_0) = (t(s_0), x(s_0)) \in R$. To fix ideas, let us assume that $s_0 \geqslant 0$ and let us set ${}^b\gamma_0 = {}^b\gamma(s) \in T^*\partial Y$. Because of the t-GCC, we may assume that ${}^b\gamma_0$ is a nondiffractive point. Note that the case $s_0 \leqslant 0$ can be treated similarly.

Let now V be an open neighborhood of (t_0, x_0) in $\mathbb{R} \times M$ such that $V_\partial = V \cap (\mathbb{R} \times \partial\Omega) \subset R$. In V, we extend the function $u(t,x)$ by zero outside $\mathbb{R} \times \Omega$ and denote this extension by $\underline{u}$. Since $u_{|V_\partial} = \partial_n u_{|V_\partial} = 0$ we observe that $\underline{u}$ solves $P\underline{u} = 0$ in V. As γ_0 is nondiffractive, the natural bicharacteristic associated with γ_0 has points outside Ω in any neighborhood of γ_0. By propagation of singularities for $\underline{u}$ we thus find that $\underline{u}$ is smooth at γ_0. Then, by propagation of singularities along the compressed generalized bicharacteristic flow (see [15, 16, 10]), we find that u is smooth at ρ. Having u smooth in $(0,T) \times \Omega$, we see that if $u \in N_T$ then $\partial_t u \in N_T$. The rest of the proof follows. $\square$

Remark 4.3. By duality, we have, as well, a boundary controllability result, as in Theorem 1.8″.

Remark 4.4. It is interesting to analyze, in this context of boundary observability, the examples of the disk and of the square studied in Section 3.

- For the disk (see Section 3.3): we set

$$\Gamma(t) = \{(r,\theta) \in [0,1] \times \mathbb{R} \quad | \ r = 1, \ vt < \theta < vt + a\}.$$

  This is the limit case of the case of Section 3.3, with $\varepsilon = 0$. With respect to Proposition 3.4, it is not true anymore that $T_0(v,a) < +\infty$ if v is chosen sufficiently large.

  The other items of Proposition 3.4, providing sufficient conditions such that $T(v,a) = +\infty$, are still valid.

- For the square (see Section 3.4): $\omega(t)$ is the translation of the segment $(0, 2a)$ along the boundary, moving anticlockwise and with constant speed v.

  With respect to Proposition 3.8, it is not true anymore that $T_0(v,a) < +\infty$ if v is large enough. The second item of Proposition 3.8, providing a sufficient condition such that $T(v,a) = +\infty$, is still valid.

We stress that, in Propositions 3.4 and 3.8, the fact that $T_0 < +\infty$ for v large enough was due to the fact that the width of the observation domain is positive. This remark shows that observability is even more difficult to realize with moving observation domains located at the boundary.

A A class of test operators near the boundary

We denote by $\Psi^m(Y)$ the space of operators of the form $R = R_{\text{int}} + R_{\tan}$ where:

- R_{int} is a classical pseudodifferential operator of order m with compact support in $\mathbb{R} \times \Omega$, that is, satisfying $R_{\text{int}} = \varphi R_{\text{int}} \varphi$ for some $\varphi \in \mathscr{C}_c^\infty(\mathbb{R} \times \Omega)$.
- $R_{\tan}$ is a classical tangential pseudodifferential operator of order m. In the local normal geodesic coordinates introduced in Section 1.3.1 such an operator only acts in the y' variables.

If $\sigma(R_{\text{int}})$ and $\sigma(R_{\tan})$ denote the homogeneous principal symbols of R_{int} and $R_{\tan}$ respectively, we observe that their restriction to $\mathrm{Char}_Y(p) \cup T^*(\mathbb{R} \times \partial\Omega)$ is well defined. Then, by means of the map $j : T^*Y \to {}^bT^*Y$, the function $\sigma(R_{\text{int}})_{|\,\mathrm{Char}_Y(P)} + \sigma(R_{\tan})_{|\,\mathrm{Char}_Y(P) \cup T^*(\mathbb{R} \times \partial\Omega)}$ yields a *continuous* map on $\hat{\Sigma} = j(\mathrm{Char}_Y(p)) \cup E$, that we denote by $\kappa(R)$. Its homogeneity yields a continuous function on $S^*\hat{\Sigma} = \hat{\Sigma}/(0, +\infty)$.

We then have the following proposition (see [11, 6]).

Proposition A.1. *Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of $H^1(\mathbb{R}\times\Omega)$ that satisfies $(\partial_t^2 - \triangle_g)u_n = 0$ and weakly converges to 0. Then, there exist a subsequence $(u_{\varphi(n)})_{n\in\mathbb{N}}$ and a positive measure μ on $S^*\hat{\Sigma}$ such that*

$$(Ru_{\varphi(n)}, u_{\varphi(n)})_{H^1(\Omega)} \xrightarrow[n\to+\infty]{} \langle \mu, \kappa(R)\rangle, \qquad R \in \Psi^0(Y).$$

This extends results in the interior introduced in the seminal works [8, 22].

References

[1] N. Anantharaman and M. Léautaud, *Sharp polynomial decay rates for the damped wave equation on the torus*, With an appendix by Stéphane Nonnenmacher, Anal. PDE **7** (2014), no. 1, 159–214.

[2] C. Bardos, G. Lebeau, and J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim. **30** (1992), no. 5, 1024–1065.

[3] N. Burq, *Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel*, Acta Math. **180** (1998), no. 1, 1-–29.

[4] N. Burq and P. Gérard, *Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes (French) [A necessary and sufficient condition for the exact controllability of the wave equation]*, C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), no. 7, 749–752.

[5] N. Burq and M. Hitrik, *Energy decay for damped wave equations on partially rectangular domains*, Math. Res. Lett. **14** (2007), no. 1, 35–47.

[6] N. Burq and G. Lebeau, *Mesures de défaut de compacité, application au système de Lamé*, Ann. Sci. École Norm. Sup. (4) **34** (2001), no. 6, 817–870.

[7] C. Castro, A. Münch, and N. Cindea, *Controllability of the linear one-dimensional wave equation with inner moving forces*, SIAM J. Control Optim. **52** (2014), no. 6, 4027–4056.

[8] P. Gérard, *Microlocal defect measures*, Comm. Partial Differential Equations **16** (1991), 1761–1794.

[9] A. Haraux, *Une remarque sur la stabilisation de certains systèmes du deuxième ordre en temps*, Portugal. Math. **46** (1989), no. 3, 245–258.

[10] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Vol. 3, Springer-Verlag (1985).

[11] G. Lebeau, *Équation des ondes amorties*, Algebraic and geometric methods in mathematical physics (Kaciveli, 1993), Math. Phys. Stud. 19, 73–109, Kluwer Acad. Publ., Dordrecht, 1996.

[12] G. Lebeau and L. Robbiano, *Contrôle exact de l'équation de la chaleur*, Comm. Partial Differential Equations **20** (1995), 335—356.

[13] J.-L. Lions, *Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués*, Tome 1, Recherches en Mathématiques Appliquées [Research in Applied Mathematics], Masson (1988).

[14] J.-L. Lions, *Exact controllability, stabilizability and perturbations for distributed systems*, SIAM Rev. **30** (1988), 1–68.

[15] R.B. Melrose and J. Sjöstrand, *Singularities of boundary value problems. I*, Comm. Pure Appl. Math. **31** (1978), no. 5, 593–617.

[16] R.B. Melrose and J. Sjöstrand, *Singularities of boundary value problems. II*, Comm. Pure Appl. Math. **35** (1982), no. 2, 129–168.

[17] K. D. Phung, *Polynomial decay rate for the dissipative wave equation* J. Differential Equations **240** (2007), no. 1, 92–124.

[18] J. Rauch and M. Taylor, *Exponential decay of solutions to hyperbolic equations in bounded domains.* Indiana Univ. Math. J. **24** (1974), 79—86.

[19] D. L. Russell, *Boundary value control of the higher-dimensional wave equation*, SIAM J. Control **9** (1971), 29–42.

[20] D. L. Russell, *Boundary value control theory of the higher-dimensional wave equation. II.*, SIAM J. Control **9** (1971), 401–419.

[21] E. Schenck, *Exponential stabilization without geometric control* Math. Res. Lett. **18** (2011), no. 2, 379–388.

[22] L. Tartar, *H-measures: a new approach for studying homogenization, oscillations and concentration effects in partial differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **115** (1990), 193–230.